The Instantaneous Screw Axis of Motions in the Kinematic Image Space

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Abstract. In the velocity analysis of mechanisms the instantaneous screw axes and the corresponding axodes play an important role. The instantaneous screw axis is computed via the velocity operator, this is the skew-symmetric matrix $\mathbf{A}\dot{\mathbf{A}}$, where $\mathbf{A}$ is the transformation matrix. From this operator the Plücker coordinates of the instantaneous screw axis are known. When the Study parameters of a one parametric motion are given a direct computation of the instantaneous screw axis would be more convenient. Without computing $\mathbf{A}$ and its derivative first, this paper shows a way of computing the instantaneous screw axis directly from the Study parameterization of the one parametric motion.

Key words: Kinematic mapping, Axodes, Plücker coordinates, velocity operator.

1 Introduction

In kinematics the velocity operator for a given motion in Euclidean three-space is a well-known concept. In some applications it is important to determine the velocity distribution and the axodes, these are the ruled surfaces representing the instantaneous screw axis in the fixed and the moving frame of one parametric motions. Let $\mathbf{A}(t)$ be the homogeneous 4×4 matrix description of such a one parametric motion in $\mathbb{E}^3$. The matrix representation of the velocity operator is then given by the skew-symmetric 4×4 matrix

$$\Omega = \mathbf{A}\dot{\mathbf{A}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \tau_x & 0 & -\omega_z & \omega_y \\ \tau_y & \omega_z & 0 & -\omega_x \\ \tau_z & -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{1}$$

The matrix representation in Eq. (1) is often rearranged to the vector notation, the so-called velocity screw $\mathbf{v} = (\omega_x, \omega_y, \omega_z, \tau_x, \tau_y, \tau_z)^\mathsf{T}$, as stated by Bottema and Roth [2] or Husty et al. [4]. Its entries determine the linear and angular velocities.

In the last centuries Study parameters, a point model for Euclidean displacements, were of great benefit in the investigation of kinematic properties of mech-

anisms [5]. In this model one parameter motions are curves in the so-called Study quadric S_6^2, which carries all points in the kinematic image space that correspond to Euclidean displacements. Recently it turned out that S_6^2 can be neglected for some problems [6], for example for motion design in P^7.

To the best of the authors knowledge until now there exists no such velocity operator in the kinematic image space, not for curves on nor off the Study quadric. Therefore one has to map the curve from P^7 back to the matrix description in $\mathbb{E}^3$ and compute the velocity screw and the axodes there.

The scope of this paper is to investigate an operator that acts on the Study parameters directly to compute the linear and angular velocities and furthermore the instantaneous screw axes. These facts are shown for some examples.

The paper is organized as follows: In Sec. 2 a brief introduction to the used notations and theories will be stated, which will be applied in Sec. 2.1 for curves in the Study quadric and Sec. 2.2 for curves not included in the Study quadric. The final Sec. 3 will show examples, such as the well-known *RPRP* and the Bennett mechanism, and finally for a motion given by a line not included in the Study quadric, which corresponds to a vertical Darboux motion.

2 Velocity Operator in Kinematic Image Space

Let the coordinates in kinematic image space P^7 be denoted by the homogeneous coordinates $(\mathbf{x}, \mathbf{y})^\mathsf{T}$, with $\mathbf{x} = (x_0, x_1, x_2, x_3)^\mathsf{T}$ and $\mathbf{y} = (y_0, y_1, y_2, y_3)^\mathsf{T}$. For the following computations $(\mathbf{x}, \mathbf{y})^\mathsf{T}$ is defined as column vector. Since [6] it is known that curves in kinematic image space correspond to Euclidean motions in the task space $\mathbb{E}^3$, nevertheless if they are in the Study quadric, which can be written as

$$x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \quad \Leftrightarrow \quad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\mathsf{T} \cdot \mathbf{y} = 0 \tag{2}$$

or off the Study quadric.

In the following $\mathbf{m}(t) = (\mathbf{p}(t), \mathbf{q}(t))^\mathsf{T}$, where $\mathbf{p}(t) = (p_0(t), \ldots, p_3(t))^\mathsf{T}$ and $\mathbf{q}(t) = (q_0(t), \ldots, q_3(t))^\mathsf{T}$, should describe a curve (or a one parameter motion) in P^7. For brevity the parameter t is avoided in the notation and $\dot{\mathbf{m}}$ denotes the derivative with respect to t. The curve $\mathbf{m}_\infty = (\mathbf{0}, \mathbf{p})^\mathsf{T}$ lies in the exceptional generator of the Study quadric, this is the three space represented by $x_0 = x_1 = x_2 = x_3 = 0$ and is connected with the original curve via the fibrization in [6]. A fiber through an arbitrary point $(x_0 : \ldots : y_3)$ outside the exceptional generator is defined by the straight line

$$(x_0 : x_1 : x_2 : x_3 : y_0 + tx_0 : y_1 + tx_1 : y_2 + tx_2 : y_3 + tx_3), \tag{3}$$

where t is the parameter of the line. The intersection points with the Study quadric correspond to the parameter values $t_1 = \infty$ and $t_2 = -\langle \mathbf{x}, \mathbf{y} \rangle / \langle \mathbf{x}, \mathbf{x} \rangle$ Note that the point of intersection with $t = \infty$ lies in the exceptional generator.

For the following inspections we use normalized coordinates, which means that $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, which is no loss of generality.

2.1 Curves in the Study Quadric

At first we restrict the curve $\mathbf{m} = (\mathbf{p}, \mathbf{q})^{\mathsf{T}}$ to be contained in the Study quadric. Then it is straight forward to compute the operator Σ by collecting the coefficients of the derivatives in the vectorial version of the velocity screw $\mathbf{A}\dot{\mathbf{A}}$. It can be written as

$$\Sigma = \begin{pmatrix} -p_1 & p_0 & -p_3 & p_2 & 0 & 0 & 0 & 0 \\ -p_2 & p_3 & p_0 & -p_1 & 0 & 0 & 0 & 0 \\ -p_3 & -p_2 & p_1 & p_0 & 0 & 0 & 0 & 0 \\ q_1 & -q_0 & q_3 & -q_2 & p_1 & -p_0 & p_3 & -p_2 \\ p_2 & -q_3 & -q_0 & q_1 & p_2 & -p_3 & -p_0 & p_1 \\ q_3 & q_2 & -q_1 & -q_0 & p_3 & p_2 & -p_1 & -p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{Q} & -\mathbf{P} \end{pmatrix} \tag{4}$$

and this yields via

$$\mathbf{v} = \Sigma \cdot \dot{\mathbf{m}} = (\omega_x, \omega_y, \omega_z, \tau_x, \tau_y, \tau_z)^{\mathsf{T}} \tag{5}$$

the velocity screw. Using the notation $\mathbf{v}_\infty = (0,0,0,\omega_x,\omega_y,\omega_z)^{\mathsf{T}}$ the Plücker coordinates of the instantaneous screw axis can be written as

$$S_I = \mathbf{v} - \frac{\langle \dot{\mathbf{p}}, \dot{\mathbf{q}} \rangle}{\langle \dot{\mathbf{p}}, \dot{\mathbf{p}} \rangle} \cdot \mathbf{v}_\infty \tag{6}$$

where the coefficient $\langle \dot{\mathbf{p}}, \dot{\mathbf{q}} \rangle / \langle \dot{\mathbf{p}}, \dot{\mathbf{p}} \rangle$ is the instantaneous pitch, which is zero for instantaneous rotations. Combining Eqs. (4), (5) and (6) yields

$$S_I = \Phi \cdot \dot{\mathbf{m}} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{Q} - \frac{\langle \mathbf{p}, \mathbf{q} \rangle}{\langle \dot{\mathbf{p}}, \dot{\mathbf{p}} \rangle}\mathbf{P} & -\mathbf{P} \end{pmatrix} \cdot \dot{\mathbf{m}}. \tag{7}$$

Equation (7) yields an operator, which computes the Plücker coordinates of the instantaneous screw axes in the fixed frame using the motion $\mathbf{m}$ and its derivative $\dot{\mathbf{m}}$, as long as $\mathbf{m}$ lies in the Study quadric. The matrix Φ is a 6×8 matrix. Geometrically S_I are the Plücker coordinates of the fixed axode. Note that S_I really represent Plücker coordinates [8], because they fulfill the Plücker relation. Using the embedding of those line coordinates of P^5 in the kinematic image space P^7 like described in [9] the moving axode can be computed with the inverse transformation.

2.2 Curves not contained in the Study Quadric

Lets consider $\mathbf{m}^* = (\mathbf{p}^*, \mathbf{q}^*)^{\mathsf{T}}$ to be a curve in $P^7 \notin S_6^2$, i.e. $\langle \mathbf{p}^*, \mathbf{q}^* \rangle \neq 0$. The derivative of $\mathbf{m}^*$ with respect to t is denoted by $\dot{\mathbf{m}}^*$. Because of the fibrization shown in Eq. (3) the curve $\mathbf{m}^*$ and its derivative $\dot{\mathbf{m}}^*$ are pulled onto S_6^2 and its tangent space, respectively, by

$$\mathbf{m} = \Pi \cdot \mathbf{m}^*, \qquad \dot{\mathbf{m}} = \overline{\Pi} \cdot \dot{\mathbf{m}}^* \tag{8}$$

where

$$\Pi = \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_4 \\ -\langle \mathbf{p}^*, \mathbf{q}^* \rangle \mathbf{I}_4 & \mathbf{I}_4 \end{pmatrix} \tag{9}$$

and

$$\overline{\Pi} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_4 \\ \langle \mathbf{p}^*, \mathbf{q}^* \rangle (2\mathbf{p}^* \cdot \mathbf{p}^{*\mathsf{T}} - \mathbf{I}_4) - \mathbf{p}^* \cdot \mathbf{q}^{*\mathsf{T}} & \mathbf{I}_4 - \mathbf{p}^* \cdot \mathbf{p}^{*\mathsf{T}} \end{pmatrix}. \tag{10}$$

Note, that $\Pi = \overline{\Pi} = \mathbf{I}_8$ if $\mathbf{m}^* \in S_6^2$. This can be computed by using the equation $\langle \mathbf{p}^*, \dot{\mathbf{q}}^* \rangle + \langle \dot{\mathbf{p}}^*, \mathbf{q}^* \rangle = 0$, which is the derivative of $\langle \mathbf{p}^*, \mathbf{q}^* \rangle$.

To compute the instantaneous screw axes S_I of the motion given by $\mathbf{m}^*$, Eq. (7) has to be applied to the projected curve $\mathbf{m}$ and the projected derivative $\dot{\mathbf{m}}$ computed in Eq. (8).

3 Examples

To illustrate this process, the fixed and the moving axode will be computed for some one parametric motions.

3.1 The RPRP Mechanism

The RPRP is a single-loop four bar mechanism with two revolute (R) and two prismatic (P) joints (see for example [3]). The motion of the coupler [7] is given by

$$\mathbf{m} = \frac{1}{\Delta} \left(0, -6t, 6, 0, 0, \sqrt{3}(12t + 5\sqrt{3}), \sqrt{3}t(12t + 5\sqrt{3}), -12(t^2 + 1) \right)^{\mathsf{T}} \tag{11}$$

where $\Delta = (-2 + \sqrt{3})/\sqrt{(7 - 4\sqrt{3})(t^2 + 1)}$, which is a curve in S_6^2. Therefore we can use the theory in Sec. 2.1 to compute

$$\Phi = \frac{1}{\Delta} \begin{pmatrix} t & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & -1 & -t & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6}\sqrt{3}\delta_1 & 0 & -2t^2 - 2 & -\frac{1}{6}\sqrt{3}\delta_2 & -t & 0 & 0 & -1 \\ \frac{1}{6}\sqrt{3}\delta_2 & 2t^2 + 2 & 0 & \frac{1}{6}\sqrt{3}\delta_1 & 1 & 0 & 0 & -t \\ -2t^2 - 2 & \frac{1}{6}\sqrt{3}\delta_2 & -\frac{1}{6}\sqrt{3}\delta_1 & 0 & 0 & 1 & t & 0 \end{pmatrix} \tag{12}$$

with $\delta_1 = (-12t^3 + 5\sqrt{3})$ and $\delta_2 = (24t^2 + 12 + 5t\sqrt{3})$. Then the instantaneous screw axis, and therefore the Plücker coordinates of the fixed axode are

$$S_I = \Phi \cdot \dot{\mathbf{m}} = (0, 0, 1, 4t, 2(t^2 - 1), 0)^{\mathsf{T}}. \tag{13}$$

The moving axode can be computed via the inverse transformation and can be written as

$$\left(0, 0, t^2 + 1, 4(t^3 - t^2 s + t s^2 + s), 2(t^2 s^2 - 3t^2 + 4ts - s^2 - 1), 0\right)^{\mathsf{T}} \qquad (14)$$

where t is the parameter of the motion and s is the parameter on the surface. Fig. 1 shows the fixed axode and some discrete copies of the moving axode (in the base frame) during the motion.
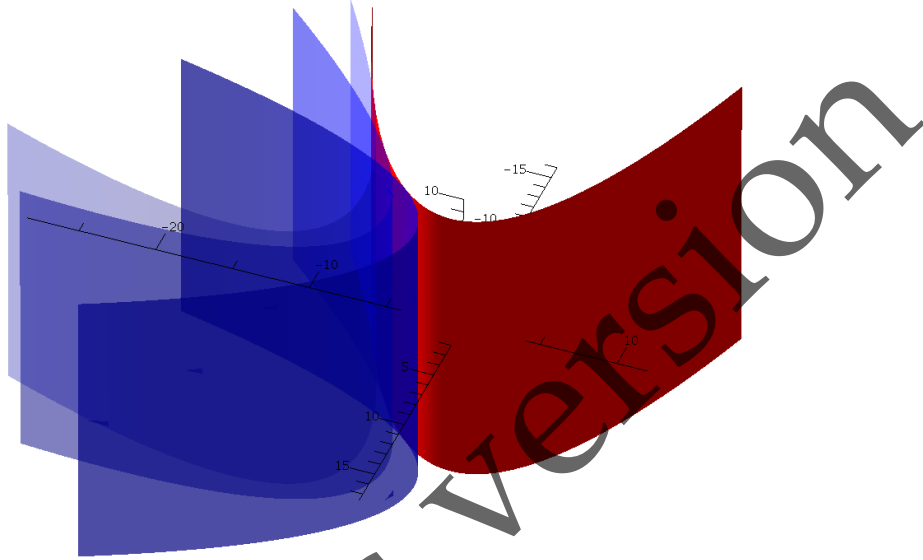

Fig. 1 Fixed axode (red) and some discrete copies of the moving axode (blue) of the RPRP

3.2 Bennett Mechanism

Despite the spherical or planar four-bar, the Bennett mechanism is the only spatial single-loop closed four bar with revolute joints only [1]. The motion of the coupler [7] is given by

$$\mathbf{m} = \frac{1}{\Delta \sqrt{t^2 + 1}} \begin{pmatrix} 0 \\ (2 - \sqrt{2})t \\ \sqrt{2}/2(t^2 - 2\sqrt{2} + 3) \\ -1/2(\sqrt{3}\sqrt{2} - 2\sqrt{3} - 2\sqrt{2} + 2)(t^2 + 1) \\ 0 \\ 3/2(\sqrt{3}\sqrt{2} - 2\sqrt{3} - 2\sqrt{2} + 2)t \\ 3/4(3\sqrt{3}\sqrt{2} - 4\sqrt{3} - 4\sqrt{2} + 6)(t^2 + 2\sqrt{2} + 3) \\ -3/4(2 - \sqrt{2})(t^2 + 1) \end{pmatrix} \qquad (15)$$

with $\Delta = \sqrt{(t^2+1)(3\sqrt{3}\sqrt{2}-4\sqrt{3}-5\sqrt{2}+8)-6\sqrt{2}+8}$ and the instantaneous screw axis is

$$S_I = \begin{pmatrix} -8(\sqrt{2}+\sqrt{3})t \\ -4(\sqrt{2}+\sqrt{3})(t^2-1) \\ -4(1+\sqrt{2})((\sqrt{2}-1)^2-t^2) \\ -6/\delta((\sqrt{3}+1)^2-2t^2)((\sqrt{2}-1)^2-t^2)t \\ -3/\delta((\sqrt{3}+1)^2-2t^2)((\sqrt{2}-1)^2-t^2)(t^2-1) \\ 3/\delta((\sqrt{3}+1)^2-2t^2)(\sqrt{3}\sqrt{2}-\sqrt{3}-\sqrt{2}+2)(t^2+1)^2 \end{pmatrix} \quad (16)$$

with $\delta = \sqrt{3}\sqrt{2}t^2+t^4-\sqrt{3}t^2+\sqrt{2}t^2-2\sqrt{3}\sqrt{2}-t^2+3\sqrt{3}-4\sqrt{2}+6$.

Although computation with the operator Φ is quite simple the expressions, also in this simple example, are too complicated to be displayed here. The corresponding axodes are plotted in Fig. 2.
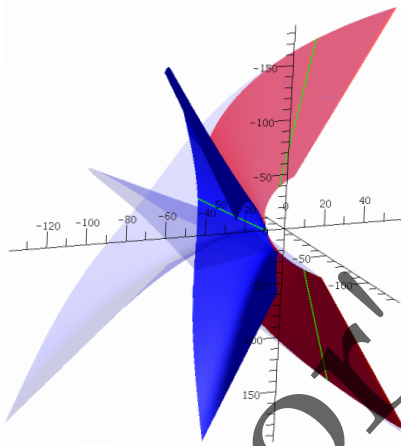

Fig. 2 Fixed axode (red) and some discrete copies of the moving axode (blue) of the Bennett
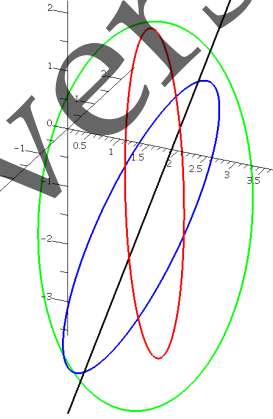
Fig. 3 Some point paths during the motion given by $\mathbf{m}^*$

3.3 Straight line in P^7

As an example for a curve not included in the Study quadric, consider the connecting line $\mathbf{m} = \mathbf{a}_1 \vee \mathbf{a}_2$ of the two arbitrarily chosen points $\mathbf{a}_1 = (5,6,7,8,13,7,9,2)^\mathsf{T}$ and $\mathbf{a}_2 = (9,3,1,7,13,5,13,17)^\mathsf{T}$. A parameterization of this line is given by

$$\mathbf{m}^*(t) = \frac{1}{\Delta}(-4t+9,3t+3,6t+1,t+7,13,2t+5,-4t+13,-15t+17)^\mathsf{T}, \quad (17)$$

with $\Delta = \sqrt{62t^2 - 28t + 140}$.

As an example of a curve not contained in S_6^2 the theory developed in Sec. 2.2 is applied. At first the line $\mathbf{m}^*$ has to be pulled to the Study Quadric using Π of Eq. (9). This yields

$$\mathbf{m} = \begin{pmatrix} (9-4t)/\Delta \\ 3(t+1)/\Delta \\ (6t+1)/\Delta \\ (t+7)/\Delta \\ -(132t^3 - 923t^2 - 1097t + 556)/\Delta^3 \\ (223t^3 + 488t^2 - 517t - 92)/\Delta^3 \\ -(50t^3 - 1221t^2 + 2463t - 1556)/\Delta^3 \\ -(897t^3 - 1750t^2 + 2525t - 532)/\Delta^3 \end{pmatrix} \qquad (18)$$

with $\Delta = \sqrt{62t^2 - 28t + 140}$.

In the second step $\dot{\mathbf{m}}^*$ has to be mapped to the tangent space of S_6^2 using $\overline{\Pi}$ of Eq. (10) to compute

$$\dot{\mathbf{m}} = \begin{pmatrix} -2(251t + 217)/\Delta^3 \\ -6(38t - 77)/\Delta^3 \\ -2(73t - 427)/\Delta^3 \\ -14(32t - 17)/\Delta^3 \\ -2(25841t^3 + 102195t^2 - 188607t - 65114)/\Delta^5 \\ -2(19811t^3 - 75468t^2 - 73257t + 38122)/\Delta^5 \\ -18(4089t^3 - 14851t^2 - 999t + 15526)/\Delta^5 \\ -2(35413t^3 + 44070t^2 - 177849t + 165578)/\Delta^5 \end{pmatrix} . \qquad (19)$$

As shown in [6] the resulting motion $\mathbf{m}$ is a vertical Darboux motion, i.e. a rotation around a fixed axis combined with a harmonic oscillation along the same axis. In this motion all point paths are ellipses, as shown in Fig. 3 for some points. Therefore the fixed and moving axodes have to be fixed lines in this example. They can be written as

$$S_I = (-1414, 53732, 36764, -2560, -70279, 102617)^\mathsf{T}. \qquad (20)$$

4 Conclusions

This article shows how to compute the instantaneous screw axis directly from curves on the Study quadric. Furthermore it was shown how to pull a curve and its derivative onto the Study quadric and its tangent space, respectively, via a fibrization of the kinematic image space.

A developed operator in this publication can be used to directly compute the axodes of a given motion in P^7.

The benefit of this work is that all the operators can be used on normalized Study parameters and there is no need to use the matrix representation of Euclidean displacements.

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