

Rational Parametrization of Linear Pentapod's Singularity Variety and the Distance to it

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Abstract. A linear pentapod is a parallel manipulator with five collinear anchor points on the motion platform (end-effector), which are connected via SPS legs to the base. This manipulator has five controllable degrees-of-freedom and the remaining one is a free rotation around the motion platform axis (which in fact is an axial spindle). In this paper we present a rational parametrization of the singularity variety of the linear pentapod. Moreover we compute the shortest distance to this rational variety with respect to a suitable metric. Kinematically this distance can be interpreted as the radius of the maximal singularity free-sphere. Moreover we compare the result with the radius of the maximal singularity free-sphere in the position workspace and the orientation workspace, respectively.

Key words: Pentapod, Kinematic Singularity, Rational Variety, Singularity-free zone.

1 Introduction

The Stewart-Gough platform (sometimes called simply Stewart platform) can be defined as a six degree-of-freedom (DOF) parallel manipulator (PM) with six identical spherical-prismatic-spherical (SPS) legs, where only the prismatic joints are active. This parallel robot is merely used in *flight simulation* where a replica cockpit plays the role of the moving platform.

Although the Stewart platform is the most celebrated PM, some of its sub-assemblies with a lower number of legs are of interest from theoretical and practical points of view. Sometimes these sub-assemblies are referred to as *components* [9]. In this paper we study the so-called *line-body component*, which is a rigid sub-assembly of a Stewart PM consisting of a linear motion platform (end-effector) named ℓ and five SPS legs, where the base anchor points can have position in \mathbb{R}^3 . Here this component is referred to as *linear pentapod*, which is an alternative to serial robots for handling axis-symmetric tools (see Fig. 1). Moreover we use the following notations:

1. The position of ℓ is given by the vector $\mathbf{p} = (p_x, p_y, p_z)^T$ and the orientation of ℓ is defined by a unit-vector $\mathbf{i} = (u, v, w)^T$.

2. The coordinate vector \mathbf{b}_j of the platform anchor point of the j th leg is described by the equation $\mathbf{b}_j = \mathbf{p} + r_j \mathbf{i}$ for $j = 1, \dots, 5$.
3. The base anchor point of the j -th leg has coordinates $\mathbf{a}_j = (x_j, y_j, z_j)^T$.

Note that all vectors are given with respect to a fixed reference frame, which can always be chosen and scaled in a way that the following conditions hold:

$$x_1 = y_1 = z_1 = y_2 = z_2 = z_3 = 0 \quad \text{and} \quad x_2 = 1. \quad (1)$$

According to [13, Theorem 12] one possible point-model for the configuration space \mathcal{C} of the linear pentapod reads as follows: *There exists a bijection between \mathcal{C} and all real points $\mathfrak{C} = (u, v, w, p_x, p_y, p_z) \in \mathbb{R}^6$ located on the singular quadric $\Gamma : u^2 + v^2 + w^2 = 1$.* Based on this notation we study the singularity loci of linear pentapods and the distance to it in the paper at hand, which is structured as follows:

We close Section 1 by a review on the singularity analysis of linear pentapods and recall the implicit equation of the singularity variety. In Section 2 we give a brief introduction to rational varieties and present a rational parametrization of the singularity loci of linear pentapods. In Section 3 we compute the minimal distance to the singularity variety with respect to a novel metric in the ambient space \mathbb{R}^6 of the configuration space \mathcal{C} . We also compute the closest singular configuration under the constraint of a fixed orientation and a fixed position, respectively. Finally a conclusion and a plan for future research is given.

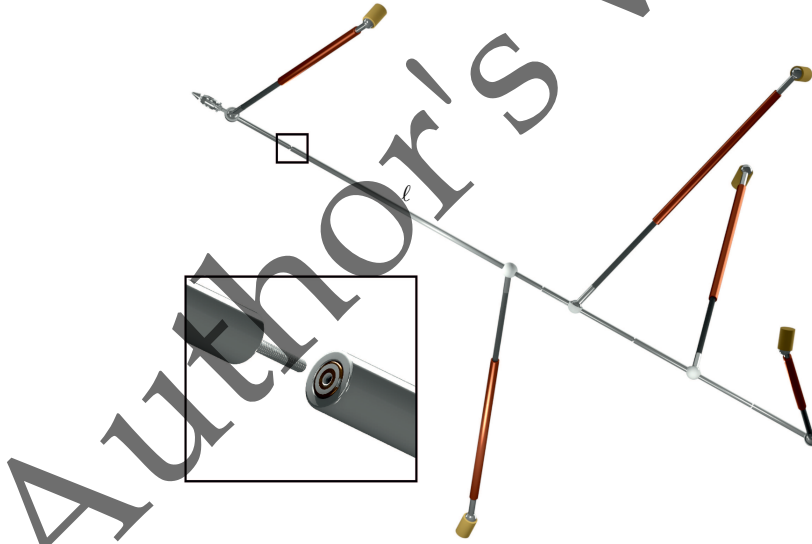


Fig. 1 Linear pentapod with the following architectural parameters: $\mathbf{a}_1 = (0, 0, 0)^T$, $\mathbf{a}_2 = (5, 0, 0)^T$, $\mathbf{a}_3 = (-4, -3, 0)^T$, $\mathbf{a}_4 = (3, 7, -6)^T$, $\mathbf{a}_5 = (9, -5, 4)^T$, $(r_1, r_2, r_3, r_4, r_5) = (0, 2, 4, 5, 10)$. Moreover it should be noted that in the illustrated design the linear platform ℓ consists of five parts, which are jointed by four passive rotational joints (a zoom of this detail is given in the box). This construction enlarges the workspace by compensating some joint limits of the platform S-joints.

1.1 Singularity Variety of the Pentapod

Singularity analysis plays an important role in motion planning of PMs. For linear pentapods the singularities as well as the singular-invariant leg-rearrangements have been studied in [5] for a planar base and in [3] for a non-planar one. A complete list of architectural singular designs of linear pentapods is given in [14], where also non-architecturally singular designs with self-motions are classified (see also [12]).

Kinematical singularities occur whenever the Jacobian matrix \mathbf{J} becomes rank deficient, where \mathbf{J} can be written as follows (cf. [3]):

$$\mathbf{J} = \begin{pmatrix} \mathbf{I}_1 & \dots & \mathbf{I}_5 \\ \hat{\mathbf{I}}_1 & \dots & \hat{\mathbf{I}}_5 \end{pmatrix}^T \quad \text{with } \mathbf{I}_j = \begin{pmatrix} p_x + r_j u - x_j \\ p_y + r_j v - y_j \\ p_z + r_j w - z_j \end{pmatrix}, \quad \hat{\mathbf{I}}_j = \begin{pmatrix} z_j(p_y + r_j v) - y_j(p_z + r_j w) \\ x_j(p_z + r_j w) - z_j(p_x + r_j u) \\ y_j(p_x + r_j u) - x_j(p_y + r_j v) \end{pmatrix}.$$

This 5×6 Jacobian matrix \mathbf{J} has a rank less than five whenever the determinants of all its 5×5 sub-matrices vanish. So by naming the determinant of the 5×5 sub-matrix, which results from excluding the j th column, with F_j the singularity loci equals $V(F_1, \dots, F_6)$; i.e. the variety of the ideal spanned by the polynomials F_1, \dots, F_6 . It can easily be checked by direct computations that this variety equals the zero-set of the greatest common divisor F of F_1, \dots, F_6 . This singularity polynomial F has the following structure:

$$\begin{aligned} F := & (A_1 p_y + A_2 p_z) u^2 + [(A_3 p_x + A_4 p_y + A_5 p_z + A_6) v + (A_7 p_x + A_8 p_y \\ & + A_9 p_z + A_{10}) w + (A_{11} p_y + A_{12} p_z) p_x + A_{13} p_y^2 + (A_{14} p_z + A_{15}) p_y \\ & + A_{16} p_z^2 + A_{17} p_z] u + (A_{18} p_x + A_{19} p_z + A_{20}) v^2 + [(A_{21} p_x + A_{22} p_y \\ & + A_{23} p_z + A_{24}) w + A_{25} p_x^2 + (A_{26} p_y + A_{27} p_z + A_{28}) p_x + (A_{29} p_z \\ & + A_{30}) p_y + A_{31} p_z^2 + A_{32} p_z] v + (A_{33} p_x + A_{34} p_y + A_{35}) w^2 + [A_{36} p_x^2 \\ & + (A_{37} p_y + A_{38} p_z + A_{39}) p_x + A_{40} p_y^2 + (A_{41} p_z + A_{42}) p_y + A_{43} p_z] w \end{aligned} \quad (2)$$

where the coefficients A_i belong to the ring $\mathbb{R} = \mathbb{R}[x_3, x_4, x_5, y_3, y_4, y_5, z_4, z_5, r_1, \dots, r_5]$ which evidently makes F a polynomial with the total-degree of 3 belonging to $\mathbb{R}[u, v, w, p_x, p_y, p_z]$. Note that for a specified orientation (u, v, w) the equation $F = 0$ determines only a quadric surface $\Omega(u, v, w)$ in the space of positions. This property is of great importance later on.

Remark 1. It can easily be checked that the polynomial F is identical with the determinant of a 7×7 matrix given in [3, Eq. (4)]. \diamond

2 Rational Parametrization of the Singularity Variety

In this section we rationally parametrize the singularity variety, which is given by the implicit equation $F = 0$. But before stepping into the computations, the presentation of a formal definition of this parametrization seems necessary.

Definition 1. Let \mathbb{K} be a field and $V \subset \mathbb{K}^m$ and $W \subset \mathbb{K}^n$ be irreducible affine varieties. A *rational mapping* from V to W is a function ϕ represented by

$$\phi : V \dashrightarrow W \quad \text{with} \quad \phi(x_1, \dots, x_m) = \left(\frac{f_1(x_1, \dots, x_m)}{g_1(x_1, \dots, x_m)}, \dots, \frac{f_n(x_1, \dots, x_m)}{g_n(x_1, \dots, x_m)} \right) \quad (3)$$

where $\frac{f_i}{g_i} \in \mathbb{K}(x_1, \dots, x_m)$ and satisfies the following properties:

1. ϕ is defined at some point of V .
2. For every $(a_1, \dots, a_m) \in V$ where ϕ is defined, $\phi(a_1, \dots, a_m) \in W$.

Definition 2. Two irreducible varieties V and W are said to be *birationally equivalent* if there exist rational mappings $\phi : V \dashrightarrow W$ and $\psi : W \dashrightarrow V$ such that $\phi \circ \psi$ and $\psi \circ \phi$ be equal to id_W and id_V respectively.

Definition 3. A *rational variety* is a variety that is *birationally equivalent* to \mathbb{K}^n .

One can find the extensive discussion of above definitions in [16, Chapters 1 and 2].

Having a rational parametrization of a variety has numerous advantages: If the coefficients of the polynomials f_i and g_i of Eq. (3) belong to \mathbb{Q} and if (x_1, \dots, x_m) is an element of \mathbb{Q}^m , then one obtains points with rational coordinates on the singularity variety [16, page 3]. This is a matter, which is of high importance to computer aided designs, as computers can calculate rational coordinates at a much faster rate.

Moreover the rationality of the singularity variety implies that it is *path connected*, which means that every singular pose can be connected to any other singular pose by a continuous singular motion [7]. This property can be used for a computationally efficient approximation of the singularity-free workspace by hierarchical structured hyperboxes, where only their boundaries have to be checked to be free of singularities. Beside the rationally parametrized singularity loci of the planar 3-RPR PM [7], only the one of Stewart PMs with planar platform and planar base [6] (see also [1, 2]) are known to the authors (in the context of PMs of Stewart-Gough type).

For the computation of the rational parametrization of the linear pentapod, we exploit the idea used in [6]: By homogenizing the singularity polynomial F of Eq. (2) by the extra variable p_0 with respect to the position variables p_x , p_y and p_z , we obtain a homogeneous polynomial $F_h \in \mathbb{R}(u, v, w)[p_x, p_y, p_z, p_0]$ in the projective 3-space \mathbb{P}^3 with homogeneous coordinates $(p_x : p_y : p_z : p_0)$. It turns out that the point B with homogeneous coordinates $(u : v : w : 0)$ is a point of the singularity variety; i.e. $B \in V(F_h) \subset \mathbb{P}^3$. Note that B is the ideal point of the linear platform ℓ with orientation vector \mathbf{i} .

The side condition on the vector $\mathbf{i} = (u, v, w)^T$ to be of unit-length, can be avoided by using the *stereographic parametrization* of the unit-sphere S^2 :

$$\mathbf{x} : (t_3, t_4) \mapsto \left(\frac{2 t_3}{t_3^2 + t_4^2 + 1}, \frac{2 t_4}{t_3^2 + t_4^2 + 1}, \frac{t_3^2 + t_4^2 - 1}{t_3^2 + t_4^2 + 1} \right). \quad (4)$$

Based on this we can parametrize the lines of the bundle \mathcal{B} with vertex B in the finite space \mathbb{R}^3 of positions with coordinates (p_x, p_y, p_z) as follows:

$$\mathcal{B} : \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = a\mathbf{x}(t_3, t_4) + t_1 \frac{\partial \mathbf{x}(t_3, t_4)}{\partial t_3} + t_2 \frac{\partial \mathbf{x}(t_3, t_4)}{\partial t_4}. \quad (5)$$

Note that the bituple (t_1, t_2) fixes the line of the bundle \mathcal{B} and the parameter a determines the point on this line. By varying $(t_1, t_2) \in \mathbb{R}^2$ and setting $a = 0$ one obtains the plane through the origin, which is orthogonal to \mathbf{i} .

Plugging $\mathcal{B}(a, t_1, t_2, t_3, t_4)$ into $F = 0$ shows that the resulting expression is only linear in a , as the ideal point B is always one of the two intersection points of a line belonging to \mathcal{B} with the quadric $\Omega(\mathbf{x}(t_3, t_4))$. By solving this linear condition we get $a(t_1, t_2, t_3, t_4)$. Now the singular configurations $\mathfrak{X} = (\xi_1, \dots, \xi_6) \in \mathbb{R}^6$ of the linear pentapod can be rationally parametrized by $(\xi_1, \xi_2, \xi_3) := \mathbf{x}(t_3, t_4)$ and

$$\begin{aligned} \xi_4 &= 2 \frac{a(t_1, t_2, t_3, t_4) t_3}{t_3^2 + t_4^2 + 1} - 2 \frac{t_1 (t_3^2 - t_4^2 - 1)}{(t_3^2 + t_4^2 + 1)^2} - 4 \frac{t_2 t_3 t_4}{(t_3^2 + t_4^2 + 1)^2}, \\ \xi_5 &= 2 \frac{a(t_1, t_2, t_3, t_4) t_4}{t_3^2 + t_4^2 + 1} - 4 \frac{t_1 t_3 t_4}{(t_3^2 + t_4^2 + 1)^2} + 2 \frac{t_2 (t_3^2 - t_4^2 + 1)}{(t_3^2 + t_4^2 + 1)^2}, \\ \xi_6 &= \frac{a(t_1, t_2, t_3, t_4) (t_3^2 + t_4^2 - 1)}{t_3^2 + t_4^2 + 1} + 4 \frac{t_1 t_3}{(t_3^2 + t_4^2 + 1)^2} + 4 \frac{t_2 t_4}{(t_3^2 + t_4^2 + 1)^2}. \end{aligned} \quad (6)$$

This parametrization covers the singular variety with exception of two low-dimensional sub-variety: A missing 3-dimensional sub-variety is defined by the denominator of $a(t_1, t_2, t_3, t_4)$. In this case the residual intersection point $\in \mathbb{R}^3$ of the line belonging to \mathcal{B} with $\Omega(\mathbf{x}(t_3, t_4))$ is not determined uniquely; i.e. the complete line belongs to $\Omega(\mathbf{x}(t_3, t_4))$. As the orientation $(0, 0, 1)$ cannot be obtained by the *stereographic parametrization*, also the 2-dimensional sub-variety $\Omega(0, 0, 1)$ is missing.

Moreover for a given singular pose $\mathfrak{X} = (\xi_1, \dots, \xi_6) \in \mathbb{R}^6$ we can trivially compute t_1, \dots, t_4 in a rational way from ξ_1, \dots, ξ_6 , thus the singularity variety is a rational one (according to the Definitions 1, 2 and 3).

3 Distance to the Singularity Variety

In singularities the number of DOFs of the mechanism changes instantaneously and becomes uncontrollable. Additionally the actuator forces can become very large and cause the break down of the platform [10]. Henceforth knowing the distance of a given pose $\mathfrak{G} = (g_1, \dots, g_6) \in \mathbb{R}^6$ from the singularity variety is of great importance.

Fixed Orientation: We ask for the closest singular configuration \mathfrak{D} having the same orientation (g_1, g_2, g_3) as the given pose \mathfrak{G} . As \mathfrak{G} and \mathfrak{D} only differ by a translation, we can define the distance between these two poses by the length of the translation vector. Therefore \mathfrak{D} has to be a pedal-point on $\Omega(g_1, g_2, g_3)$ with respect to the point (g_4, g_5, g_6) . The set \mathcal{O} of all these pedal-points equals the variety $V(\frac{\partial L}{\partial p_x}, \frac{\partial L}{\partial p_y}, \frac{\partial L}{\partial p_z}, \frac{\partial L}{\partial \lambda})$ where λ is the Lagrange multiplier of the Lagrange equation

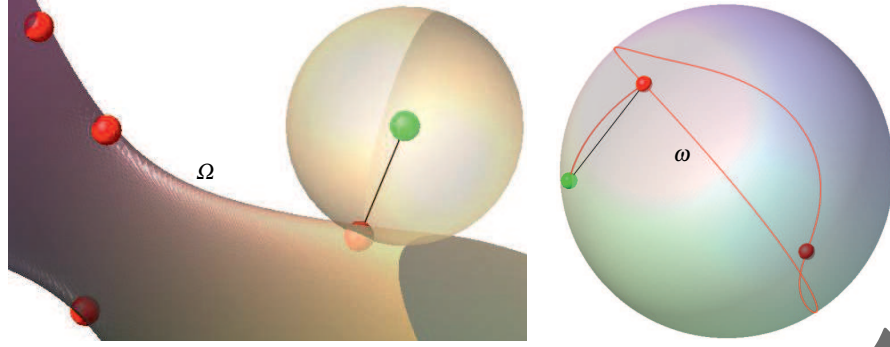


Fig. 2 Illustrations are done for $\mathcal{G} = (\frac{3}{5}, \frac{4}{5}, 0, 2, 3, 4)$ of the linear pentapod displayed in Fig. 1. Fixed orientation (Left): \mathcal{O} has only four real solutions where the closest one $\mathcal{D} = (\frac{3}{5}, \frac{4}{5}, 0, 2.5517, 2.6374, 0.1144)$ has a distance of 3.9412 units. Fixed position (right): \mathcal{P} has only two real solutions where the closest one $\mathfrak{P} = (0.3701, 0.5523, 0.7468, 2, 3, 4)$ has a spherical distance of 48.4178° .

$$L(p_x, p_y, p_z, \lambda) = (p_x - g_4)^2 + (p_y - g_5)^2 + (p_z - g_6)^2 + \lambda F. \quad (7)$$

It is well known (see [15, Appendix A]) that in general \mathcal{O} consists of six points over \mathbb{C} , where the closest one to (g_4, g_5, g_6) implies \mathcal{D} (see Figs. 2 and 3).

Fixed Position: Now we ask for the closest singular configuration \mathfrak{P} , which has the same position (g_4, g_5, g_6) as the given pose \mathcal{G} . As \mathcal{G} and \mathfrak{P} only differ in orientation, the angle $\in [0, \pi]$ enclosed by these two directions can be used as distance function. Note that this angle is the spherical distance function on S^2 .

By intersecting the singularity surface for the given position (g_4, g_5, g_6) with S^2 we obtain a spherical curve $\omega(g_4, g_5, g_6)$ of degree 4. Then \mathfrak{P} has to be a spherical pedal-point on $\omega(g_4, g_5, g_6)$ with respect to the point $(g_1, g_2, g_3) \in S^2$ (see Fig. 2). By replacing the underlying spherical distance by the Euclidean metric of the ambient space \mathbb{R}^3 , one will not change the set \mathcal{P} of pedal-points on $\omega(g_4, g_5, g_6)$ with respect to (g_1, g_2, g_3) . Therefore \mathcal{P} can be computed as the variety $V(\frac{\partial L}{\partial u}, \frac{\partial L}{\partial v}, \frac{\partial L}{\partial w}, \frac{\partial L}{\partial \lambda_1}, \frac{\partial L}{\partial \lambda_2})$ where λ_1 and λ_2 are the Lagrange multipliers of the Lagrange equation

$$L(u, v, w, \lambda_1, \lambda_2) = (u - g_1)^2 + (v - g_2)^2 + (w - g_3)^2 + \lambda_1 F + \lambda_2 G \quad (8)$$

with $G = u^2 + v^2 + w^2 - 1$. It can easily be proved (see [15, Appendix B]) that in general \mathcal{P} consists of 8 points over \mathbb{C} , where the one with the shortest spherical distance to (g_4, g_5, g_6) implies \mathfrak{P} (see Fig. 3).

Remark 2. For the practical application of this spherical distance to the singularity, we recommend to locate the position vector \mathbf{p} in the tool-center-point of ℓ . \diamond

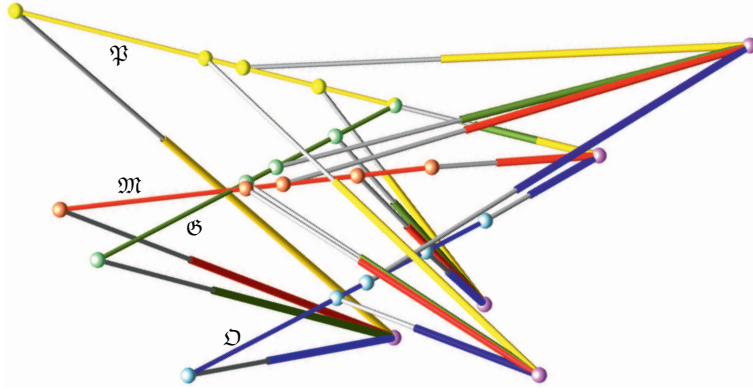


Fig. 3 Comparison of the different configurations \mathfrak{G} (green), \mathfrak{D} (blue), \mathfrak{P} (yellow) and the red-colored $\mathfrak{M} = (0.5559, 0.7274, 0.4021, 2.2966, 3.4794, 1.8357)$ with $d(\mathfrak{M}, \mathfrak{G}) = 1.4791$. In contrast $d(\mathfrak{D}, \mathfrak{G}) = 3.9412$ and $d(\mathfrak{P}, \mathfrak{G}) = 4.4142$. For this example only 16 out of 80 pedal-points are real.

General Case: In contrast to the two special cases discussed above, the general case deals with mixed (translational and rotational) DOFs, thus the question of a suitable distance function arises. As the configuration space \mathcal{C} equals the space of oriented line-elements, we can adopt the object dependent metrics discussed in [13] for our mechanical device as follows:

$$d(\mathcal{L}, \mathcal{L}')^2 := \frac{1}{5} \sum_{j=1}^5 \|\mathbf{b}_j - \mathbf{b}'_j\|^2 \quad (9)$$

where \mathcal{L} and \mathcal{L}' are two configurations and \mathbf{b}_j and \mathbf{b}'_j denote the coordinate vectors of the corresponding platform anchor points. Note that the ambient space \mathbb{R}^6 (of \mathcal{C}) equipped with the metric d of Eq. (9) is a Euclidean space (cf. [13]).

With respect to this metric d we can compute the closest singular configuration \mathfrak{M} to \mathfrak{G} in the following way: We determine the set \mathcal{M} of pedal-points on the singularity variety with respect to \mathfrak{G} as the variety $V(\frac{\partial L}{\partial u}, \frac{\partial L}{\partial v}, \frac{\partial L}{\partial w}, \frac{\partial L}{\partial p_x}, \frac{\partial L}{\partial p_y}, \frac{\partial L}{\partial p_z}, \frac{\partial L}{\partial \lambda_1}, \frac{\partial L}{\partial \lambda_2})$ where λ_1 and λ_2 are the Lagrange multipliers of the Lagrange equation

$$L(u, v, w, p_x, p_y, p_z, \lambda_1, \lambda_2) := d(\mathfrak{M}, \mathfrak{G})^2 + \lambda_1 G + \lambda_2 F. \quad (10)$$

Random examples (see [15, Appendix C]) indicate that \mathcal{M} consists of eighty points over \mathbb{C} , where the one with the shortest distance d to \mathfrak{G} equals \mathfrak{M} (see Fig. 3).

Remark 3. Note that these minimal distances can be seen as the radii of maximal singularity-free hyperspheres [10] in the position workspace (see also [11]), the orientation workspace (see also [8]) and the complete configuration space. Moreover the distance $d(\mathfrak{M}, \mathfrak{G})$ to the singularity variety can also be interpreted as quality index thus it is an alternative to the value of F proposed in [4]. \diamond

4 Conclusions and future research

We presented a rational parametrization of the singularity variety of linear pentapods in Section 2 and computed the distance to it in Section 3 with respect to the novel metric given in Eq. (9), which can easily be adopted for e.g. Stewart PMs as well. As this distance is of interest for many tasks (e.g. quality index for path planning, radius of the maximal singularity-free hypersphere, ...) a detailed study of it (e.g. efficient computation of \mathfrak{M} , proof of $\#\mathcal{M} = 80, \dots$) is dedicated to future research.

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