Application of Intersection Theory to Singularity and Motion Mode Analysis of Mechanisms

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Abstract. Different motion modes of mechanisms often correspond to irreducible components of the configuration space (c-space), and singularities of the c-space often (but not necessarily) happen at the intersections of irreducible components, i.e. motion modes, of the configuration space. Frequently mechanisms are required to perform different tasks on different motion modes of the mechanism *connected by c-space singularities*. This means that in order for the mechanism to switch between motion modes it has to pass through a c-space singularity. Although singularities may not be avoided, it is desirable to design the mechanism in such a way that the transition motion through the singularity is as smooth as possible. In this paper we propose using the theory of intersections of algebraic varieties as a tool from algebraic geometry that allows investigating this situation. Modern computational algebra provides the necessary algorithms. The theory and its implications are demonstrated for two simple examples.

1 Introduction

Most mechanisms comprise 'algebraic joints', i.e. kinematic pairs whose geometric constraints can be described by *polynomial equations* $f_1 = 0, \dots, f_k = 0$. These generate the *constraint ideal* $\mathscr{I} = \langle f_1, \dots, f_k \rangle \subset \mathbb{A}$ in the polynomial ring $\mathbb{A}$. Thus the c-space is an *algebraic variety* $\mathsf{V}(\mathscr{I})$, and algebraic geometry and commutative algebra provide a framework for the analysis, and potentially the design, of the c-space. The goal of this paper is to recall the relevant concepts from algebraic geometry facilitating the analysis of c-space singularities of mechanisms and robots. This gives us computational tools to analyze c-space singularities.

In general singularities are not desirable since the differential mobility of a mechanism changes impairing their stability and making their control difficult. However if a mechanism contains closed loops and is designed to go from one motion mode to another it has to go through a singularity. Also the simulation of the mechanism dynamics becomes difficult since standard numerical integration methods for differential algebraic (DAE) system can not handle singularities. In fact these types of mechanisms and their constraints are often used when testing and comparing different DAE-solvers [7]. If a mechanism is to perform several tasks where each one corresponds to irreducible components/motion modes V_i of the c-space variety V, then the mechanism must pass through a c-space singularity. It is thus important that the singularities and their nature are known a priori. An interesting question in this regard is whether there are tangential intersections of motion modes that allow for smooth transitions between different modes [1, 12].

To study these questions we propose to use the concept of multiplicity. If at a point in the intersection of two varieties the multiplicity of intersection is greater than one, then at the intersection there are at least some common tangent direction to both varieties, and hence a smooth transition between different modes is possible at least in principle. From an engineering perspective it would be desirable to be able to design regular/tangential intersections of mechanisms performing tasks in several motion modes that are connected by singularities since then the mechanism would not have to stop at singularity in order to change to another motion mode. Another interesting aspect about multiplicity is that it may provide a good quantitative model of what has been intuitively called the shakiness of the mechanism. This aspect will be treated more thoroughly in a forthcoming paper. A similar approach is used also in [14]. The singularity analysis is therefore particularly important, but at the same time also a difficult area of mechanism design [2, 3, 8, 13, 11, 16]. The advantage of using algebraic geometry instead of the differential geometric approach is that in algebraic geometry one can obtain global as well as local results.

Computations in this paper were performed with the program `Singular` [10].

2 Mathematical Preliminaries

2.1 Rings, Ideals and Singularities

We recall some basic facts and refer to [5, 9] for more details. The polynomial ring with coefficient field $\mathbb{K}$ and variables x_i is denoted by $\mathbb{A} = \mathbb{K}[x_1, \ldots, x_n]$. The following facts about ideals $\mathscr{I} \subset \mathbb{A}$ in $\mathbb{A}$ are fundamental

(i) Every ideal is *finitely generated*, i.e. it has a basis with a finite number of generators.
(ii) Every radical ideal can be decomposed to a finite number of prime ideals: This gives the decomposition of the variety into *irreducible components*:

$$\mathsf{V}(\mathscr{I}) = \mathsf{V}(\sqrt{\mathscr{I}}) = \mathsf{V}(I_1) \cup \cdots \cup \mathsf{V}(I_s).$$

Definition 2.1 (Tangent space) *Let $\mathscr{I} = \langle f_1, \ldots, f_k \rangle$ be an ideal and let us denote by $f = (f_1, \ldots, f_k)$ the map defined by the generators. The differential (or Jacobian) of f is then denoted by df, and its value at q is df_q. Let us suppose that $\mathscr{I}$ is a prime ideal. Then the tangent space of the variety $V = \mathsf{V}(\mathscr{I})$ at q is*

$$T_qV = \{z \in \mathbb{K}^n \mid df_q z = 0\}. \tag{1}$$

Note that by definition T_qV is a vector space, so its dimension is well defined.

Definition 2.2 (Singular points) *A point $q \in V$ is* singular *if* $\dim(T_qV) > \dim_q(V)$. *Otherwise the point q is* regular. *The set of singular points of V is denoted by $\Sigma(V)$.*

Recall that $\Sigma(V)$ is itself a variety whose dimension is less than $\dim(V)$. Hence almost all points of a variety are regular.

Throughout the paper the constraint ideal (the ideal associated with the constraints) is assumed to be prime.

Let $V = V_1 \cup \cdots \cup V_\ell$ be the decomposition to irreducible components. Then there are basically two ways of a point q of a general variety V to be singular: either q is a singularity of an irreducible component V_i or it is an intersection point of two components. That is, the variety of singular points is

$$\Sigma(V) = \bigcup_{i=1}^{\ell} \Sigma(V_i) \cup \bigcup_{i \neq j} V_i \cap V_j. \tag{2}$$

Once we have the irreducible decomposition it is easy to compute the intersections (the second term in (2)). To compute the singular points of irreducible components (the first term in (2)) one needs the concept of *Fitting ideals* [9].

Let M be a matrix of dimension $k \times n$ with entries in $\mathbb{A}$. The ℓth Fitting ideal of M, $\mathsf{F}_\ell(M)$, is the ideal generated by the $\ell \times \ell$ minors of M. Let now $f = (f_1, \ldots, f_k) : \mathbb{K}^n \mapsto \mathbb{K}^k$ be a map corresponding to the prime ideal $\mathscr{I} = \langle f_1, \ldots, f_k \rangle$ and let $V = \mathsf{V}(\mathscr{I})$ be the corresponding irreducible variety. Let us suppose that $\mathsf{dim}(V) = n - \ell$.

Theorem 2.1 (Jacobian criterion) *The singular variety of V is*

$$\Sigma(V) = \mathsf{V}\big(\mathscr{I} + \mathsf{F}_\ell(df)\big) = \mathsf{V}\big(\mathscr{I}\big) \cap \mathsf{V}\big(\mathsf{F}_\ell(df)\big).$$

In particular if $\mathscr{I} + \mathsf{F}_\ell(df) = \mathbb{A}$ then V is smooth.

One can now ask how does the variety 'look like' locally. Unlike manifolds varieties don't have to be smooth or even locally Euclidean. If q is a smooth point then naturally the tangent space T_qV can be thought of as the best local approximation. In case of singular points we need the concept of *tangent cone* [4, 15]. Let us suppose that q is the origin. Then each polynomial $f \in \mathsf{I}(V)$ can be written as a sum of its homogeneous components. Let us denote by $f_{(q,min)}$ the component of lowest degree.

Definition 2.3 (Tangent cone) *Suppose that $\mathsf{V}(\mathscr{I}) \subset \mathbb{R}^n$ is an algebraic variety and $\mathscr{I} = \langle h_1, \ldots, h_l \rangle$ and let $q \in V$. The* Tangent cone *of V at q, denoted by $C_q(V)$, is the variety*

$$C_q(V) = \mathsf{V}(f_{(q,min)} \mid f \in \mathsf{I}(V)). \tag{3}$$

The tangent cone has the following basic properties:

- q is a smooth point if and only if $C_qV = T_q\mathsf{V}$
- $\mathsf{dim}(C_q\mathsf{V}) = \mathsf{dim}_q(\mathsf{V})$.

The tangent cone is the *simplest possible approximation* of V in the neighborhood of q and thus $\mathsf{dim}(C_q\mathsf{V}) = \mathsf{dim}_q(\mathsf{V})$. The generators of the tangent cone can be obtained efficiently using Gröbner basis techniques.

2.2 Intersection Multiplicity and Singular Points

Let q be a point of a subvariety $V \subset \mathbb{K}^m$. We denote by $\mathscr{O}_{V,q}$ the local ring of V at q. Consider an ideal $\mathscr{I} \subset \mathscr{O}_{V,q}$ with $\dim_{\mathbb{K}}(\mathscr{O}_{V,q}/\mathscr{I}) < \infty$. Then the Hilbert-Samuel function of $\mathscr{I}$ is

$$H_{\mathscr{I}}(n) = \dim_{\mathbb{K}}(\mathscr{O}_{V,q}/\mathscr{I}^n) \quad (n \in \mathbb{N}). \tag{4}$$

It is known that there exists a polynomial $P_{\mathscr{I}}(n)$ such that $H_{\mathscr{I}}(n)$ and $P_{\mathscr{I}}(n)$ coincide for large n. This polynomial is called the Hilbert-Samuel polynomial of I. It is of degree $d = \dim \mathscr{O}_{V,q}$. The leading coefficient is $e(\mathscr{I})/d!$, where $e(\mathscr{I})$ is an integer. We say that $e(\mathscr{I})$ is the Hilbert-Samuel multiplicity of $\mathscr{I}$.

Let $V_1, V_2 \subset \mathbb{K}^n$ be subvarieties. Then one can define the intersection multiplicity of V_1 and V_2 at an irreducible component of the intersection $V_1 \cap V_2$. We restrict to the special case where the irreducible component is a point $q \in V_1 \cap V_2$. The definition is based on the idea of 'reduction to the diagonal' [6]. Let us define the diagonal embedding and the ideal corresponding to it:

$$\Delta : \mathbb{K}^n \to \mathbb{K}^n \times \mathbb{K}^n \quad , \quad (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, a_1, \ldots, a_n)$$
$$\delta = (x_1 - y_1, \ldots, x_n - y_n) \subset \mathbb{K}[x, y] \quad , \quad \mathsf{V}(\delta) = \Delta(\mathbb{K}^n)$$

In this way we have an isomorphism $V_1 \cap V_2 \cong \Delta(\mathbb{K}^n) \cap (V_1 \times V_2)$. We now consider δ as an ideal of the local ring $\mathscr{O}_{V_1 \times V_2, (q,q)}$.

Definition 2.4 (Intersection multiplicity) *The intersection multiplicity of V_1 and V_2 at q is*

$$i_q(V_1, V_2) = e(\delta)\,. \tag{5}$$

A fundamental theorem connecting multiplicities of points of V to its tangent cone says that the tangent cone C_qV and the variety V have the same multiplicity at q. The tangent cone gives the geometric picture of the variety near its singular points.

Gröbner basis techniques can be extended to local rings, hence the multiplicity can actually be computed. The drawback with this approach is that the number of variables is artificially doubled, making the computations potentially very time consuming. However, it is possible to do the computations in such a way that this inconvenience is avoided. This will be explored in another paper.

3 Examples

3.1 Slider-crank mechanism with circular constraints

Figure 1a) shows a planar slider-crank mechanism whose slider is constrained to move on the union of two circles with radius $1/2$ and centers at $(1,1/2)$ and $(1,-1/2)$, respectively (This can also be regarded as a planar 2R serial manipulator that must perform motion on the circles). The constraint equations are $p_i = 0, i = 1\ldots 5$ where

$$
\begin{aligned}
&p_1 = c_1 + c_2 - x,\ p_2 = s_1 + s_2 - y,\ p_3 = c_1^2 + s_1^2 - 1 = 0,\ p_4 = c_2^2 + s_2^2 - 1\\
&p_5 = ((x-1)^2 + (y-1/2)^2 - 1/4)((x-1)^2 + (y+1/2)^2 - 1/4)
\end{aligned}
$$

where $s_i := \sin x_i, c_i := \cos x_i$. The equation $p_5 = 0$ restricts the slider to move on the circles. Analyzing the constraint ideal $\mathscr{I} = \langle p_1, p_2, p_3, p_4, p_5\rangle$ yields

$$\mathscr{I} = \sqrt{\mathscr{I}} = \mathscr{I}_1 \cap \mathscr{I}_2 \subset \mathbb{Q}[c_1, s_1, c_2, s_2, x, y]$$

The singular points of $\mathsf{V}(\mathscr{I})$ are the intersections of two modes $\mathsf{V}(\mathscr{I}_1) = V_1$ and $\mathsf{V}(\mathscr{I}_2) = V_2$ which represent the motion where the end effector is constrained to move on either circles. Checking the Jacobian criterion proves that both modes, $\mathsf{V}(\mathscr{I}_1)$ and $\mathsf{V}(\mathscr{I}_2)$, are smooth. The singular points are thus

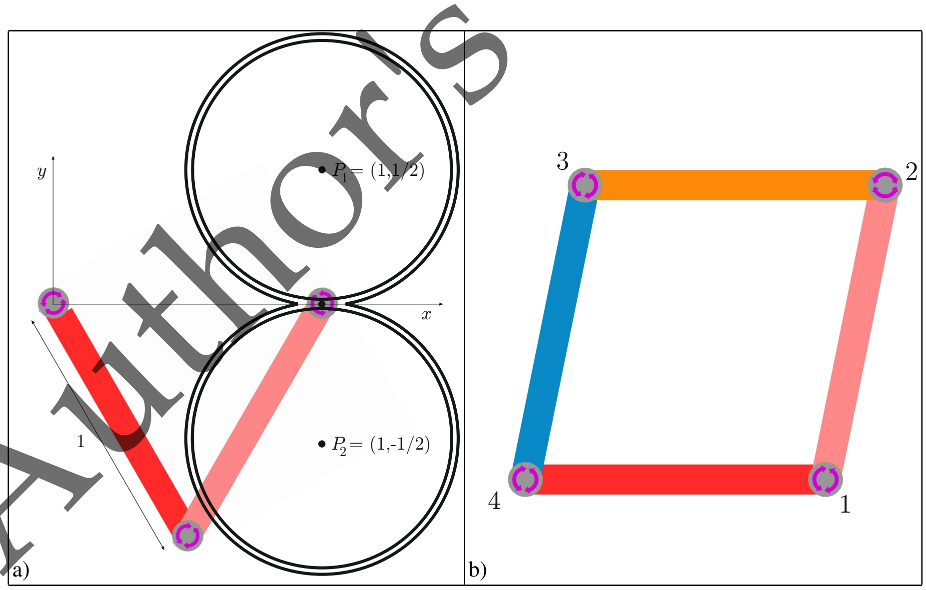

Fig. 1 a) Slider-Crank mechanism with circular constraints. b) 4-bar linkage with equal link lengths

$$\begin{aligned}\Sigma(V) &= \mathsf{V}(\mathscr{I}_1 \cap \mathscr{I}_2) = \mathsf{V}(\mathscr{I}_1 + \mathscr{I}_2)\\ &= \mathsf{V}(y, x-1, 4s_2^2-3, 2c_2-1, s_1+s_2, c_1+c_2-1) = q_+ \cup q_-\end{aligned}$$

where $q_\pm = (1/2, \mp\sqrt{3}/2, 1/2, \pm\sqrt{3}/2, 1, 0)$. Now we make a coordinate transformation so that q_+ is the origin and then double the number of variables. In this way the ideals corresponding to $\mathscr{I}_1$ and $\mathscr{I}_2$ to can be written as

$$\begin{aligned} i_1 &= \langle b_1+b_2-y_1, a_1+a_2-z_1, y_1^2+z_1^2-y_1,\\ &\quad 2b_2y_1+2a_2z_1+2a_2-(\sqrt{3}+1)y_1-z_1, a_2^2+b_2^2+a_2-\sqrt{3}b_2\rangle\\ i_2 &= \langle e_1+e_2-y_2, d_1+d_2-z_2, y^2+z_2^2+y_2,\\ &\quad 2e_2y_2+2d_2z_2+2d_2-(\sqrt{3}-1)y_2-z_2, d_2^2+e_2^2+d_2-\sqrt{3}e_2\rangle\end{aligned}$$

Hence the sum $\mathscr{J} = i_1 + i_2$ is an ideal in the ring

$$\mathbb{A} = \mathbb{Q}(\sqrt{3})[a_1, b_1, a_2, b_2, d_1, e_1, d_2, e_2, z_1, y_1, z_2, y_2]$$

Let us now consider the quotient ring $\mathbb{A}/\mathscr{J}$ corresponding to the variety $V = \mathsf{V}(\mathscr{J})$. This construction is needed when one does actual computations in the local ring $\mathscr{O}_{V,q}$. Hence the ideal in definition 2.4 is now also interpreted as

$$\delta = \langle a_1-d_1, b_1-e_1, a_2-d_2, b_2-e_2, y_1-y_2, z_1-z_2\rangle \subset \mathbb{A}/\mathscr{J}.$$

The Hilbert polynomial of δ is now $P_\delta(n) = 1+2n$ and hence the multiplicity is $i_{q_+}(\mathsf{V}_1, \mathsf{V}_2) = 2$. Since V_1 and V_2 are smooth *we must have tangential intersection* of modes V_1 and V_2 at q_+.

The tangent cone at q_+ can be computed as

$$C_0V = \mathsf{V}(a_1+a_2-z, b_1+b_2-y, 2a_2-z-\sqrt{3}y, 6b_2-\sqrt{3}z-3y, y^2) = (L_1)^2$$

which is a 'doubled' tangent line $y^2 = 0$ in the plane

$$T = \mathsf{V}(6b_2-\sqrt{3}z, 2a_2-z, 6b_1+\sqrt{3}z, 2a_1-z),$$

and indicates the same result, i.e. that the multiplicity is two.

3.2 Four-Bar mechanism with equal link lengths

The constraint equations for the mechanism in figure 1b) can be formulated by using joint 4 as cut-joint as

$$\begin{aligned} &p_4 = c_1+c_2+c_3-1 = 0,\ p_5 = s_1+s_2+s_3 = 0\\ &p_i = c_i^2+s_i^2-1,\ \ i = 1,2,3.\end{aligned}$$

Analyzing the constraint ideal $\mathscr{I} = \langle p_1, p_2, p_3, p_4, p_5 \rangle$ leads to

$$\mathscr{I} = \sqrt{\mathscr{I}} = \mathscr{I}_1 \cap \mathscr{I}_2 \cap \mathscr{I}_3 \subset \mathbb{Q}[c_1, s_1, c_2, s_2, c_3, s_3].$$

The singular points are again intersections of three smooth motion modes $\mathsf{V}(\mathscr{I}_1)$, $\mathsf{V}(\mathscr{I}_2)$ and $\mathsf{V}(\mathscr{I}_3)$. The singular points then consists of three points:

$$\Sigma(V) = \mathsf{V}(\mathscr{I}_1 \cap \mathscr{I}_2 \cap \mathscr{I}_3) = \mathsf{V}(\mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3) = \{q_0, q_1, q_2\}.$$

The mechanism is, for example, analyzed at the point

$$q_0 = (1, 0, -1, 0, 1, 0) = \mathsf{V}(\mathscr{I}_1) \cap \mathsf{V}(\mathscr{I}_2) = V_1 \cap V_2.$$

A coordinate transformation such that q_0 is the origin yields the ideals

$$\begin{aligned} i_1 &= \langle a_2^2 + b_2^2 - 2a_2, b_3, a_3, b_1 + b_2 + b_3, a_1 + a_2 + a_3 \rangle \\ i_2 &= \langle d_3^2 + e_3^2 + 2e_3, d_2 + d_3, e_2 + e_3, d_1 + d_2 + d_3, e_1 + e_2 + e_3 \rangle, \end{aligned}$$

corresponding to $\mathscr{I}_1$ and $\mathscr{I}_2$ after the transformation. As above the sum $\mathscr{J} = i_1 + i_2$ is considered in the ring

$$\mathbb{A} = \mathbb{Q}[a_1, b_1, a_2, b_2, a_3, b_3, d_1, e_1, d_2, e_2, d_3, e_3]$$

and then we need to consider the quotient ring $\mathbb{A}/\mathscr{J}$. Computing as before we obtain in this case $i_{q_0}(\mathsf{V}_1, \mathsf{V}_2) = 1$. Hence the intersection is not tangential in this case as expected. If we compute the tangent cone at q_0 we get

$$\begin{aligned} C_{q_0}V &= \mathsf{V}(a_1 + a_2 + a_3, b_1 + b_2 + b_3, 2a_2, 2a_3, 2b_2b_3 + 2b_3^2) \\ &= \mathsf{V}(a_1, b_1 + b_2, a_2, a_3, b_3) \cup \mathsf{V}(a_1, b_1, b_2, a_3, b_2 + b_3) = L_1 \cup L_2 \subset \mathbb{R}^6 \end{aligned}$$

Now L_1 and L_2 clearly represent two *different* lines in $\mathbb{R}^6$:

$$L_1 \cup L_2 = \{t(0, -1, 0, 1, 0, 0) \mid t \in \mathbb{R}\} \cup \{t(0, 0, 0, 1, 0, -1) \mid t \in \mathbb{R}\}$$

The lines intersect at a nonzero angle which implies that multiplicity is one.

4 Conclusion

The ability to treat the c-spaces of mechanisms and robots as algebraic varieties has many advantages. Most importantly essential properties, like singularities and mobility, can be algorithmically computed and possibly designed. In this paper we have introduced the concept of intersection multiplicity in order to investigate the 'order of tangency' of intersections of different motion modes of a mechanism. This is demonstrated for two simple examples. The presented method has been applied to larger systems as well that cannot be presented here. It was observed that the complexity scales up well. Tangential intersections in particular are of practical importance since a mechanism could transit regularly between motion modes, i.e. not have to stop when switching between motion modes, which also reduces constraint forces.

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