A Study on Constraints Violation in Dynamic Analysis of Spatial Mechanisms

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Abstract. The main goal of this work is present a comparative study on several methodologies to solve the equations of motion of constrained spatial multibody systems taking into account the problem of constraints violation. In the sequel of this process, the two main categories of methods to eliminate or reduce constraints violation are revisited, namely those that are based on constraint stabilization approaches and direct correction formulations. Particular attention is given to the most popular approaches, that is, Baumgarte stabilization formulation, penalty method, Augmented Lagrangian formulation and a direct correction approach. Finally, several examples of application are considered to compare the accuracy and efficiency of the different methods used throughout this study.

Key words: Constraints violation, Baumgarte method, Penalty approach, Augmented Lagrangian formulation, Direct correction, Multibody dynamics.

1 Introduction

By and large, the methods to deal with the problem of constraints violation for dynamics of constrained multibody mechanical systems can be divided into three main groups, namely: (*i*) constraint stabilization approaches; (*ii*) coordinate partitioning methods and (*iii*) direct correct formulations [1]. The constraint stabilization approaches are the most popular due to their simplicity and easiness for computational implementation [2]. The coordinate partitioning methods have the great merit of allowing the rigorous resolution of the constraint equations at the position, velocity and acceleration levels. However, they suffer from poor numerical efficiency due to the requirement for the iterative solution for dependent generalized coordinates in the Newton-Raphson method [3]. Finally, the direct formulations have physical meaning, computational efficiency, but they can exhibit some numerical instability [4].

The main focus of this study, which closely follows the recent work by Flores and his coauthors [1], is on the elimination of the constraints violation in dynamic analysis of spatial mechanisms. For this, body coordinates formulation is utilized to describe the system components and the kinematic joints. The equations governing the dynamic behavior of the systems incorporate corrective terms that are added to the position and velocity vectors in order to satisfy the corresponding constraint equations. These corrective terms are expressed in terms of the Jacobian matrix and kinematic constraint equations. The corrective terms are added and considered during the numerical resolution of the dynamic equations of motion. Results for spatial mechanisms are presented and utilized to discuss the assumptions and procedures adopted throughout this work.

2 Methods to Handle Constraints Violation

The translational and rotational equations of motion for dynamic analysis of constrained spatial mechanisms can be expressed in the form [5]

$$\begin{bmatrix} \mathbf{M} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{1}$$

Applying any method suitable for the resolution of linear algebraic equations can solve this linear system of equations. The existence of null elements in the main diagonal of the leading matrix and the possibility of ill-conditioned matrices suggest that methods using partial or full pivoting are preferred. In a simple way, Eq. (1) is solved for the accelerations then, the velocities and positions can be obtained by numerical integration. This procedure must be repeated until the final time of analysis is reached. This manner to solve the dynamic equations of motion is commonly referred to as the standard Lagrange multipliers method [5].

It is known that Eq. (1) does not use explicitly the position and velocity equations associated with the kinematic constraints. Consequently, during the simulations, the constraint equations start to be violated. In order to keep the constraint violations under control, the Baumgarte stabilization method can be considered [2]. This method allows constraints to be slightly violated before corrective actions can take place, in order to force the violation to vanish. Thus, using the Baumgarte approach, the equations of motion for a system subjected to kinematic constraints can be stated in the following form

$$\begin{bmatrix} \mathbf{M} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} - 2\boldsymbol{\alpha}\dot{\boldsymbol{\Phi}} - \boldsymbol{\beta}^2\boldsymbol{\Phi} \end{Bmatrix} \tag{2}$$

If α and β are chosen as positive constants, the stability of the general solution of Eq. (2) is guaranteed. Baumgarte [2] highlighted that the suitable choice of the parameters α and β can be performed by numerical experiments. Hence, the Baumgarte method has some ambiguity in determining optimal feedback gains. The improper choice of these parameters can lead to unacceptable results in the dynamic analysis of the multibody systems [6].

The penalty method constitutes an alternative way to solve the dynamic equations of motion. In this method, the equations of motion are modeled as a linear second-order differential equation that can be written in the form [7]

$$m_c\ddot{\mathbf{\Phi}} + d_c\dot{\mathbf{\Phi}} + k_c\mathbf{\Phi} = \mathbf{0} \tag{3}$$

Taking into account the second derivative of the algebraic constraint equations, then Eq. (3) yields

$$m_c(\mathbf{D}\dot{\mathbf{v}} + \dot{\mathbf{D}}\mathbf{v}) + d_c\dot{\mathbf{\Phi}} + k_c\mathbf{\Phi} = \mathbf{0} \tag{4}$$

Pre-multiplying Eq. (4) by the transpose of Jacobian matrix, $\mathbf{D}^T$, and after mathematical treatment, results in

$$m_c\mathbf{D}^T\mathbf{D}\dot{\mathbf{v}} = -\mathbf{D}^T(m_c\dot{\mathbf{D}}\mathbf{v} + d_c\dot{\mathbf{\Phi}} + k_c\mathbf{\Phi}) \tag{4}$$

Let consider now the Newton-Euler equations of motion for a system of unconstrained system and written here as [5]

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{g} \tag{5}$$

Summation of Eqs. (4) and (5), and after some basic mathematical manipulations yields

$$(\mathbf{M} + \alpha\mathbf{D}^T\mathbf{D})\dot{\mathbf{v}} = \mathbf{g} - \alpha\mathbf{D}^T(-\gamma + 2\mu\omega\dot{\mathbf{\Phi}} + \omega^2\mathbf{\Phi}) \tag{6}$$

where

$$\alpha = m_c, \quad d_c = 2\mu\omega m_c \quad \text{and} \quad k_c = \omega^2 m_c \tag{7}$$

Equation (7) is solved for the accelerations. This method gives good results if α tends to infinity. Typical values of α, ω and μ are 1×10^7, 10 and 1, respectively [9]. It should be noted that with this penalty method, multibody systems with redundant constraints or kinematic singular configurations could be easily solved

The augmented Lagrangian formulation penalizes the constraints violation, in the same form as the Baumgarte stabilization method. This is an iterative procedure that presents a number of advantages relative to other methods because it involves the solution of a smaller set of equations, handles redundant constraints and still delivers accurate results in the vicinity of singular configurations [7]. The augmented Lagrangian formulation consists of solving the system equations of motion by an iterative process. Let index i denote the i-th iteration. The evaluation of the system accelerations in a given time step starts as

$$\mathbf{M}\dot{\mathbf{v}}_i = \mathbf{g}, \quad (i = 0) \tag{8}$$

The iterative process to obtain the accelerations proceeds with the evaluation of the following equtions obtaining the accelerations

$$(\mathbf{M} + \alpha\mathbf{D}^T\mathbf{D})\dot{\mathbf{v}}_{i+1} = \mathbf{M}\dot{\mathbf{v}}_i - \alpha\mathbf{D}^T(-\gamma + 2\mu\omega\dot{\mathbf{\Phi}} + \omega^2\mathbf{\Phi}) \tag{9}$$

This iterative process continues until

$$\left\|\dot{\mathbf{v}}_{i+1} - \dot{\mathbf{v}}_i\right\| = \varepsilon \tag{10}$$

where ε is a specified tolerance.

In what follows, an approach to deal with the elimination of the constraints violation at both position and velocity levels is briefly described [1]. For this, let consider that during the numerical resolution of the dynamic equations of motion, the vector of generalized coordinates needs to be corrected due to the constraints violation. Thus, the corrected positions can be expressed in the form

$$\mathbf{q}^c = \mathbf{q}^u + \mathrm{d}\mathbf{q} \tag{11}$$

where $\mathbf{q}^u$ denotes the uncorrected positions and $\delta\mathbf{q}$ is the set of corrections that eliminates the constraints violation. This means that the corrective term has to be added to vector $\mathbf{q}^u$ in order to ensure that the constraint equations are satisfied, i.e.

$$\boldsymbol{\Phi}(\mathbf{q}^c) = \boldsymbol{\Phi}(\mathbf{q}^u) + \delta\boldsymbol{\Phi} = \mathbf{0} \tag{12}$$

The term $\delta\boldsymbol{\Phi}$ in Eq. (12) can be understood as the variation of the constraint equations and can be expressed as [8]

$$\delta\boldsymbol{\Phi} = \frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{q}_1}\delta\mathbf{q}_1 + \frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{q}_2}\delta\mathbf{q}_2 + \ldots + \frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{q}_n}\delta\mathbf{q}_n = \mathbf{D}\delta\mathbf{q} \tag{13}$$

Combining now Eqs. (12) and (13) yields

$$\boldsymbol{\Phi}(\mathbf{q}^u) + \mathbf{D}\delta\mathbf{q} = \mathbf{0} \tag{14}$$

which ultimately leads to

$$\delta\mathbf{q} = -\mathbf{D}^{-1}\boldsymbol{\Phi}(\mathbf{q}^u) \tag{15}$$

In general, the Jacobian matrix, $\mathbf{D}$, is not square, therefore, $\mathbf{D}^{-1}$ does not exist. However, the concept of the Moore-Penrose generalized inverse matrix, $\mathbf{D}^{+}$, can be employed as [1]

$$\mathbf{D}^{+} = \mathbf{D}^T(\mathbf{D}\mathbf{D}^T)^{-1} \tag{16}$$

such that

$$\mathbf{D}\mathbf{D}^{+}\mathbf{D} = \mathbf{D} \qquad \mathbf{D}^{+}\mathbf{D}\mathbf{D}^{+} = \mathbf{D}^{+} \tag{17}$$

and both $\mathbf{D}^{+}\mathbf{D}$ and $\mathbf{D}\mathbf{D}^{+}$ are symmetric matrices. Consequently, it is possible to establish the following mathematical relation [1]

$$\mathbf{D}^T(\mathbf{D}\mathbf{D}^T)^{-1} = \mathbf{D}^T(\mathbf{D}^{+})^T\mathbf{D}^{+} = (\mathbf{D}^{+}\mathbf{D})^T\mathbf{D}^{+} = \mathbf{D}^{+}\mathbf{D}\mathbf{D}^{+} = \mathbf{D}^{+} \tag{18}$$

Thus, Eq. (15) can be rewritten in the following form

$$\delta\mathbf{q} = -\mathbf{D}^T(\mathbf{D}\mathbf{D}^T)^{-1}\boldsymbol{\Phi}(\mathbf{q}^u) \tag{19}$$

Finally, introducing Eq. (19) into Eq. (14) yields

$$\mathbf{q}^c = \mathbf{q}^u - \mathbf{D}^T(\mathbf{D}\mathbf{D}^T)^{-1}\boldsymbol{\Phi}(\mathbf{q}^u) \tag{20}$$

that represents the corrected generalized coordinates in each integration time step. It must be noticed that the kinematic constraint equations at the position level are, in general, nonlinear, then Eq. (20) must be solved iteratively by employing a numerical algorithm, such as the Newton-Raphson method.

A similar analysis can be performed at the velocity level, resulting in

$$\mathbf{v}^c = \mathbf{v}^u - \mathbf{D}^T(\mathbf{D}\mathbf{D}^T)^{-1}\dot{\mathbf{\Phi}}(\mathbf{q}^c,\mathbf{v}^u) \tag{21}$$

that represents the corrected generalized velocities in each integration time step.

The described methodology can be easily incorporated in the standard method to solve the dynamic equations of motion. The approach described above does not consider weighting factors to the coordinates and velocities variables. In order to take into account different weighting factors, some works have been proposed to include inertia of bodies, which allow for the adjustments to be made in an inverse manner to the system inertia. The basic idea of this approach is that the more massive bodies are moved the least if the constraints allow that [8].

3 Results and Discussion

In order to examine the effectiveness of the approaches briefly presented in the previous section, a spatial four bar mechanisms is considered as an example of application. Figure 1 depicts the initial configuration of this mechanism, which includes three moving bodies, a non-moving body (the ground), two revolute joints and two spherical joints. The revolute joint that connects the crank to the ground is along the *x*-axis, while the revolute joint that connects the follower to the ground is in the *xy* plane and makes a 45º angle with the *y*-axis. At the initial time, the crank is along the *z*-axis and the other two moving bodies are in the *yz* plane. A spring element is also considered in this multibody system model in which the spring stiffness and the natural length are equal to 50 N/m and 0.8 m, respectively. Governing properties of the four bar mechanism are presented in Table 1. The initial conditions necessary to characterize this multibody model are obtained from a kinematic analysis for an input constant crank speed equal to 2π rad/s.

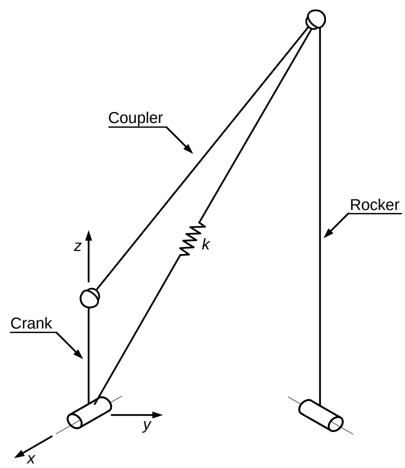

Fig. 1 Spatial four bar mechanism modeled

Table 1. Governing properties for the four bar linkage

Body	Length [m]	Mass [kg]	Moment of inertia [kgm^2]		
			$I_{\xi\xi}$	$I_{\eta\eta}$	$I_{\zeta\zeta}$
Crank	0.020	0.50	0.03	0.03	0.03
Coupler	0.064	1.50	0.02	0.02	0.02
Rocker	0.070	0.15	0.02	0.02	0.02

Long time computational simulations are performed to test and compare the accuracy and efficiency of use different methods to solve the dynamic equations of motion. For this purpose, five approaches are considered, namely the standard method based on the technique of Lagrange multipliers, the Baumgarte method, the penalty method, the augmented Lagrangian formulation and the described methodology. The quantitative measure of the efficiency of these approaches is drawn from the constraints violation as $\mathbf{\Phi}^T\mathbf{\Phi}$, as well as from the computation time of the dynamic simulations. Table 2 gives the parameters used for the different models, necessary to characterize the problem.

Table 2. Parameters used for the dynamic simulations

Final time of simulation	5.0 s	Baumgarte - α	5
Integrator algorithm	ode45	Baumgarte - β	5
Reporting time step	0.02 s	Penalty - α	1×10^{7}
Relative tolerance	1×10^{-6}	Penalty - ω	10
Absolute tolerance	1×10^{-9}	Penalty - μ	1

Figure 2 shows the constraints violation at the position level for the four bar mechanism. It should be noticed that different scales are used for the results plotted in Figs. 2a-b, in order to clearly observe the effect of the method used to solve the system dynamics on the constraints violation. By analyzing the diagrams of Figs. 2, it can be observed that when the standard method is utilized the violation of the constraint equations grows indefinitely with time. As it was expected, the standard method based on the Lagrange multipliers technique produces unacceptable results because the kinematic constraint equations are rapidly violated due to the inherent errors and instability that develop during computations. With the stabilization methods, the behavior of the different methods is significantly different, in the measure that the level of the constraints violation is kept under control during the dynamic simulations. Indeed, with the Baumgarte approach, the penalty method and the augmented Lagrangian formulation experience tells that the numerical results do not diverge from the exact solution, but oscillate around it. Magnitude and frequency of the oscillations depend on the values of the penalty parameters used. Finally, when the described methodology is utilized to solve the dynamic equations of motion, the constraints violation is eliminated as it can be observed in Figs. 2. In fact, with the described approach the average of the constraints violation is of order 1.0×10^{-16}.

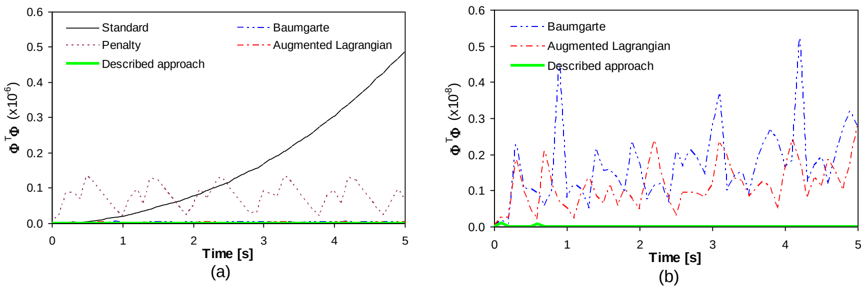

Fig. 2 Constraints violation ($\Phi^T\Phi$) of the four bar mechanism

Figure 3 depicts the computation time consumed in dynamic simulations for the four bar mechanism, which can be used to have an idea about the computational efficiency of the different methods used to solve the system dynamics. The most efficient method to deal with the constraints violation is the Baumgarte approach. It can be observed that the described approach does not penalize the total amount of computation time when compared with the other methods to solve the dynamic equations of motion. It must be stated thet the standard method is, in fact, the most efficient approach, however, it does not take into account the problem associated with the constraints violation.

The efficiency of the described method can be understood by its nature, in the measure that the two additional blocks are added to the standard solution of the equations of motion [1]. The elimination of the constraints violation for positions needs an iterative scheme, because the corrective terms are dependent on the positions. However, based on the computational tests performed, this process requires at most three iterations to eliminate the constraints violation at the position to an acceptable level. The constraints violation for velocities are eliminated with a single step, since constraints at the velocity level are linear and the corrective terms are computed as function of the corrected positions performed previously.

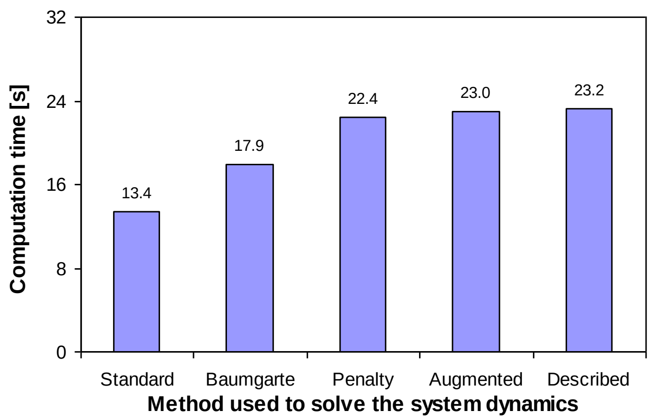

Fig. 3 Computation time for the four bar mechanism

6 Conclusions

A comparative study on the several methodologies to handle the problem of constraints violation in forward dynamics of constrained spatial mechanical systems has been presented in this work. For this, the most commonly used method to deal with resolution of the equations of motion and constraints violation have been revisited, namely the Baumgarte stabilization method, the penalty approach and the augmented Lagrangian formulation. In addition, an alternative approach to eliminate the violation of the kinematic constraint equations in the framework of forward dynamics of constrained multibody systems has been described. The basic idea of the described approach is to add corrective terms to the position and velocity vectors with the intent to satisfy the corresponding kinematic constraint equations. These corrective terms are evaluated as function of the Moore-Penrose generalized inverse of the Jacobian matrix and of the kinematic constraint equations. Finally, a spatial four bar mechanism has been considered as a demonstrative example of application to show that the effectiveness of the several approaches utilized in this study. From the obtained results, it can be drawn that the described approach is effective in eliminate the constraints violation at both positions and velocities levels without penalizing the computational efficiency.

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References

1. Marques, F., Souto, A.P., Flores, P., On the constraints violation in forward dynamics of multibody systems. Multibody System Dynamics, 39(4), 385-419 (2017)
2. Baumgarte, J., Stabilization of constraints and integrals of motion in dynamical systems. Computer Methods in Applied Mechanics and Engineering, 1, 1-16 (1972)
3. Wehage, R.A., Haug, E.J., Generalized coordinate partitioning for dimension reduction in analysis of constrained systems. Journal of Mechanical Design, 104, 247-255 (1982)
4. Fisette, P., Vaneghem, B., Numerical integration of multibody system dynamic equations using the coordinate partitioning method in an implicit Newmark scheme. Computer Methods in Applied Mechanics and Engineering, 135, 85-105 (1996)
5. Nikravesh, P.E., Computer-Aided Analysis of Mechanical Systems, Prentice Hall, Englewood Cliffs, New Jersey (19889
6. Flores, P., Machado, M., Seabra, E., Silva, M.T., A parametric study on the Baumgarte stabilization method for forward dynamics of constrained multibody systems. Journal of Computational and Nonlinear Dynamics, 6(1), 0110191-9 (2011)
7. Jalón, J.G., Bayo, E., Kinematic and Dynamic Simulations of Multibody Systems: The Real-time Challenge. Springer Verlag, New York (1994)
8. Flores, P., A methodology for quantifying the position errors due to manufacturing and assemble tolerances. Journal of Mechanical Engineering, 57(6), 457-467 (2011)