Constraint equations of inverted kinematic chains

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Abstract. A lot of different kinematic chains have been investigated focusing on constraint equations, singularities, assembly modes and motion capabilities. The approach to obtain constraint equations via inverted chains however is rarely considered. We provide a detailed look on the constraint varieties of inverted chains, beginning with the basics of quaternion conjugation. The transformation of the Denavit- Hartenberg parameters needed for the quaternion conjugation is discussed in the paper. The quaternion conjugation is a fast way to obtain the variety corresponding to the inverted kinematic chain. Geometrically the conjugation is a reflection in the kinematic image space $\mathbb{P}^7$ with respect to a line and a five- dimensional subspace. Some examples of constraint equations of kinematic chains and their inverted chains complete the paper.

Key words: Kinematic chains, inverted kinematic chains, constraint equations, dual quaternion conjugation

1 Introduction

In recent years the description of kinematic chains and parallel or serial mechanisms by systems of polynomial equations has become more and more popular because of its success in describing the direct and inverse kinematics, a global analysis of singularities, workspaces and operation modes (see e.g. [4], [10], [11],[6]). The main reason for this success is the availability of more and more sophisticated algebraic manipulation systems that can deal with large systems of polynomial equations, the advance in the implementation of algorithms developed in algebraic geometry in such systems, but also the advances in the global numerical solution methods of these equations (see e.g. [13]). Therefore it makes sense to search for polynomial descriptions of all thinkable kinematic chains with the goal to use these descriptions in the synthesis and analysis of mechanisms and robots that are designed by combinations of different kinematic chains. This is the goal of a joint research program between IRCyNN Nantes and the University of Innsbruck. In the course of this project the question arose how the constraint equations of a kinematic chain change when base and endeffector are interchanged. A quick - but wrong- answer to this problem would be that one obtains the inverse of the motion and this yields the same motion. The mistake in this assumption can be seen immediately when one studies simple planar motions like the Cardan and its inverse, the Oldham motion. It is well known that the degree of these two motions differ. How the two simple motions are linked was already discussed in [1]. That inverted parallel manipulators behave quite differently was recently shown in [6].

When displacements are described in Study parameters or dual quaternions, then it is well known that the inverse displacement can be described by simple conjugation of the dual quaternion. The effect of inverting the motion of a mechanism - like a parallel manipulator - on its set of polynomial constraint equations is less known. It is the goal of this paper to shed some light on this issue with the basic idea to simplify the necessary work in finding the constraint equations of inverted kinematic chains. In [15], the so called implicitization algorithm was developed, which always can be used to derive the constraint equations of a kinematic chain. This algorithm essentially computes the implicit constraint equations by eliminating the motion parameters of the classical forward kinematics of the chain. It is obvious that this algorithm could be applied to the inverted chain to obtain the constraint equations. But it is desirable to avoid this computationally laborious algorithm whenever possible. It will be shown explicitly in this paper that quaternion conjugation of the set of constraint equations yields the same result as the implicitization algorithm.

The paper is organized as follows: In Section 2 the geometric interpretation of quaternion conjugation in the kinematic image space will be discussed, in Section 3 the effect on the design parameters (Denavit-Hartenberg parameters) of the chain is studied, which is necessary to compare the implicitization algorithm with the conjugation of the set of constraint equations. In Sections 4 and 5 the theory is applied to $RP-$ (revolute-prismatic-chains) and $PR-$ as well as RRP- and PRR-chains.

2 Conjugation of Quaternions

In the following 3D-Euclidean displacements (SE(3)) are described in a point model, which is obtained via kinematic mapping[1]. In this mapping every Euclidean displacement corresponds to a point with homogeneous projective coordinates $(x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3)$ located on the Study quadric $S_6^2 : x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0$, which is a six dimensional quadric in a seven dimensional projective space $\mathbb{P}^7$. This space is called kinematic image space, sometimes also Soma space. The projective point coordinates can also be interpreted as the components of a dual quaternion. Both interpretations will be used simultaneously in the following.

The scope of this chapter is to investigate the constraint equations of an inverted serial chain without using the implicitization algorithm but with the use of conjugate quaternions. Before this can be done some basic properties of kinematic chains

[1] Due to space limitation this mapping cannot be explained in detail, but a comprehensive introduction can be found in [4] or [3].

and the image space $\mathbb{P}^7$ must be recalled. The kinematics of a serial chain is described with respect to (arbitrarily chosen) coordinate frames in the base and the end-effector. All possible locations of the end-effector with respect to the base correspond to algebraic varieties which are described by sets of polynomial equations in $\mathbb{P}^7$. Coordinate transformations in the base and the end-effector frame induce linear mappings T in $\mathbb{P}^7$ that preserve several interesting geometric objects:

1. the *Study quadric* S_6^2,
2. the *Null cone* defined by $\mathscr{N}: x_0^2+x_1^2+x_2^2+x_3^2=0$, which is quadric in $\mathbb{P}^7$, that has only complex points with exception of its 3-dimensional vertex space $\mathscr{E}: x_0=x_1=x_2=x_3=0$. $\mathscr{E}$ is entirely contained in S_6^2 and is called *exceptional generator space*,
3. the *exceptional quadric* $\mathscr{Y}: y_0^2+y_1^2+y_2^2+y_3^2=0 \in \mathscr{E}$,
4. all quadrics $\mathscr{Q}=\lambda S_6^2+\mu\mathscr{N}, \lambda, \mu \in \mathbb{R}$ in the pencil spanned by the Study quadric and the Null cone.

A detailed derivation and proofs for these statements and some interesting examples can be found in [9]. The invariant objects essentially govern the kinematics of 3D-Euclidean displacements[2]. The mapping of a dual quaternion with components $[x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3]$ to $[x_0,-x_1,-x_2,-x_3,y_0,-y_1,-y_2,-y_3]$ implies a mapping in $\mathbb{P}^7$ which does not effect the line $(s:0:0:0:t:0:0:0)$ with projective parameters s and t nor the five-dimensional subspace $(0:t_1:t_2:t_3:0:t_5:t_6:t_7)$ with projective parameters t_i. These properties can be shown as follows: The conjugation of a quaternion corresponds to a linear mapping, more precisely a collineation, in $\mathbb{P}^7$ described with the matrix

$$\mathbf{A}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad (1)$$

The characteristic equation $(1-\lambda)^2(-1-\lambda)^6=0$ of $\mathbf{A}$ yields the double eigenvalue $\lambda_1=1$ and a sixfold eigenvalue $\lambda_2=-1$. The corresponding eigenspaces v $(\lambda_1=1)$ and w $(\lambda_2=-1)$ are simply found to be

$$v=\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}t+\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}s, \quad w=\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}t_1+\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}t_2+\ldots+\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}t_6, \text{with } t,s,t_1,t_2,\ldots t_6 \in \mathbb{R} \qquad (2)$$

[2] The kinematic images of planar and spherical displacements subordinate completely to this description because both cases are obtained by three dimensional sub-spaces of $\mathbb{P}^7$. The corresponding geometry of their image spaces and the algorithms to derive these geometries can be found in [1] p.393ff. resp. [5] p. 60ff.

Eq. 2 shows that v can be characterized by the relations $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = 0$ which define a line in $\mathbb{P}^7$ and the second eigenspace w is given by $x_0 = y_0 = 0$ which is a five- dimensional subspace of $\mathbb{P}^7$. The intersection of the line v and S_6^2 yields two characteristic points, namely the point $I = (1:0:0:0:0:0:0:0)$ which correponds to the the identity in SE(3) and an ideal point $I_d = (0:0:0:0:1:0:0:0)$ in the exceptional space $\mathscr{E}$. It is obvious that the identity is fixed under quaternion conjugation. The line v is a fiber in the fiber projectivity φ defined in [9] and in [8], that can be used to define a non-injective “extended kinematic map”. The second eigenspace w is the span of the six points $P_1 = (0:1:0:0:0:0:0:0), \ldots, P_6 = (0:0:0:0:0:0:0:1)$. Intersecting w with S_6^2 yields the quadric $\mathscr{P}: x_1y_1 + x_2y_2 + x_3y_3 = 0$. Points on this quadric are characterized by the equations $x_0 = y_0 = 0$ and define displacements which have the property that the inverse of the displacement is the same as the displacement itself. These displacements are well known in the kinematics, Study ([14],p.178) calls them “Umwendungen” (π-turns). In the one parametric case (curves on $\mathscr{P}$) the corresponding motions are called line-symmetric motions and have been studied by Krames synthetically (see [1], Ch.9 § 7) and analytically in [12].

The planar kinematic mapping and also the effect of conjugation was developed in [1] (CH.XI, §14). To show how the planar case fits into the theory developed above, lets have a brief look into this case. Planar displacements are characterized by the equations $x_2 = x_3 = y_0 = y_1 = 0$. The intersection of these four hyperplanes yields a three-dimensional subspace $E \subset \mathbb{P}^7$, which is a generator space of S_6^2. The intersection of E with the Null cone $\mathscr{N}$ yields the quadric $x_0^2 + x_1^2 = 0$ which can be factorized into

$$V_1 : x_0 + ix_1 = 0, \qquad V_2 : x_0 - ix_1 = 0, \qquad i \ldots \text{complex unit.} \tag{3}$$

V_1 and V_2 are two complex conjugate 2-planes intersecting in a real line u. On the other hand this line intersects the exceptional quadric in two complex conjugate points $J_1 = (0:0:1:i)$ and $J_2 = (0:0:1:-i)$. Alternatively one could provide the following arguments: According to the dimension formula ($dim(v \cap E) = dim(v) + dim(E) - dim(\mathbb{P}^7) = 5 + 3 - 7 = 1$) the 5-dim subspace w and the 3-space E must intersect in a one dimensional linear subspace which is of course the line u. The intersection of the fixed space v (Eq.(2)), the Study quadric and the three-dimensional subspace T yields an intersection point, which is exactly the origin $[1,0,0,0]$ of the planar displacements. Applying the linear map

$$[x_0, x_1, y_2, y_3] \mapsto [x_0, -x_1, -y_2, -y_3] \tag{4}$$

to the 2-planes $x_0 \pm ix_1 = 0$ one can see that they are interchanged $x_0 \mp ix_1 = 0$ fixing their intersection line u. The linear map (Eq.(4)) also does not effect the origin. The map is a reflection into the origin. These results are exactly the same as in the classical planar case but within the bigger setting of spatial kinematic mapping ([1], p.433).

3 Denavit-Hartenberg Parameters

The investigation of inverse kinematic chains and a comparison of their constraint equations obtained by different methods requires an adaption of their Denavit-Hartenberg parameters when base and end-effector coordinate systems are interchanged. This change has to be observed when one wants to compare the constraint equations obtained by simple dual quaternion conjugation and the constraint equations obtained by applying the implicitization to the forward kinematics equations of the inverted chain. A simple considerations shows that the transformation can be written as

$$T_{DH}: \quad \begin{array}{ll} a_{0+j} \rightarrow -a_{n-j} & a, d, \alpha \ldots \text{DH- parameters,} \\ d_{0+j} \rightarrow -d_{n-j} & j < n \ldots \text{number of the joint to be changed} \\ \alpha_{0+j} \rightarrow -\alpha_{n-j} & n \ldots \text{total number of joints} \end{array} \qquad (5)$$

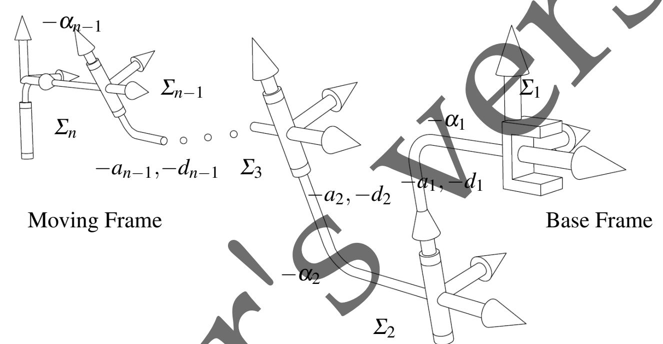

Fig. 1 Interchanged Frames [7, p. 49]

Interchanging base and end-effector frames and applying the transformation T_{DH} on the DH- parameters one can now apply quaternion conjugation and thereby obtain the constraint equations of the inverse kinematic chain having the former end-effector frame as base frame. It is easy to see that this general method also can be used when a manipulator is composed of several (even different) kinematic chains. In the next sections this method will be used to obtain the constraint equations of some kinematic chains and compare it with the results of the linear implicitization algorithm (LIA).

4 RP- and the PR-chains

To show that LIA and quaternion conjugation followed by the T_{DH} transformation yield the same solution set (the same variety), the constraint equations for the RP- and the inverse PR-chain are computed at first via LIA. For both chains the Gröbner bases of the ideal corresponding to the set of constraint equations are computed and denoted by B_1 and B_2. Onto one set of constraint equations, for example B_1, the quaternion conjugation and T_{DH} are applied resulting in a set of equations $\bar{B}_1$. Then one has to show that the variety represented by B_2 is the same as the one of $\bar{B}_1$. Note that the ideals (the set of equations) do not necessarily have to be identical but the represented varieties have to be identical. To check if the same variety is represented by the two different ideals, the radical membership has to be checked. According to Cox, Little and O'Shea [2] the necessary and sufficient condition can be formulated,

> *"Let k be an arbitrary field and let $I = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$ be an ideal. Then $f \in \sqrt{I}$ if and only if the constant polynomial 1 belongs to the ideal $\tilde{I} = \langle f_1, \ldots, f_s, 1 - yf \rangle \subset k[x_1, \ldots, x_n, y]$ (in which case, $\tilde{I} = k[x_1, \ldots, x_n, y]$)."*

This algorithm is applied to the ideals B_2 and $\bar{B}_1$. Taking a polynomial f_1 of one basis, adding $1 - yf_1$ to the second basis and computing the Gröbner bases of the ideal yields in general a remainder. If there is no remainder the polynomial f_1 is already included in the first basis. This has to be done with all polynomials of the first basis and then the same procedure is applied reversely. As expected the computation shows that the varieties of the RP- and the conjugated PR-chain are identical because no single remainder shows up. Although the equations describing the variety of the RP-chain as well as the conjugated set of constraints describing the PR-chain are simple, they still differ in some signs, as it can be seen in Eqs.(6) and (7). But the application of the radical membership test shows immediately that they are describing the same variety.

$$\begin{aligned} Cm_{RP-con} = [&(\alpha_1{}^2 a_1 - a_1)\, x_3 - 2 y_2 - 2\,\alpha_1\, y_3,\, (\alpha_1{}^2 a_1 - a_1)\, x_2 - 2\,\alpha_1\, y_2 - 2\,\alpha_1{}^2 y_3, \\ &(\alpha_1{}^2 a_1 - a_1)\, x_1 + 2\,\alpha_1{}^2 y_0 - 2\,\alpha_1\, y_1,\, (\alpha_1{}^2 a_1 - a_1)\, x_0 - 2\,\alpha_1\, y_0 + 2 y_1, \\ &\alpha_1\, y_0{}^2 + (-\alpha_1{}^2 - 1)\, y_0\, y_1 + \alpha_1\, y_1{}^2 + \alpha_1\, y_2{}^2 + (\alpha_1{}^2 + 1)\, y_2\, y_3 + \alpha_1\, y_3{}^2] \end{aligned} \tag{6}$$

$$\begin{aligned} Cm_{PR} = [&(\alpha_1{}^2 a_1 - a_1)\, x_3 + 2 y_2 - 2\,\alpha_1\, y_3,\, (\alpha_1{}^2 a_1 - a_1)\, x_2 - 2\,\alpha_1\, y_2 + 2\,\alpha_1{}^2 y_3, \\ &(\alpha_1{}^2 a_1 - a_1)\, x_1 - 2\,\alpha_1{}^2 y_0 - 2\,\alpha_1\, y_1,\, (\alpha_1{}^2 a_1 - a_1)\, x_0 - 2\,\alpha_1\, y_0 - 2 y1, \\ &\alpha_1\, y_0{}^2 + (\alpha_1{}^2 + 1)\, y_0\, y_1 + \alpha_1\, y_1{}^2 + \alpha_1\, y_2{}^2 + (-\alpha_1{}^2 - 1)\, y_2\, y_3 + \alpha_1\, y_3{}^2] \end{aligned} \tag{7}$$

5 RRP- and the PRR-chains

The investigation of the relation between the RRP- and its inverse, the PRR-chain is not as straightforward as in the previous section. The used computer system Maple is not able to compute the radical membership for the RRP- and the PRR-chain in general coordinates. But it is possible to provide an alternative, remarkably simple algorithm. The LIA comes up in both cases with a set of nine constraint equations, which are quadratic in x_i and y_i. These sets are denoted by H_1 and H_2. Four out of the nine equations are linear in y_i and are used to solve for these parameters. Because of lack of space only one of the resulting y_i is displayed, all the others look similar

$$y_0 = -\frac{a_1(\alpha_1\alpha_2-1)(\alpha_1\alpha_2+1)x_1{}^2}{2\alpha_1(\alpha_2{}^2+1)x_0} - \frac{(\alpha_1{}^2\alpha_2 a_1 - \alpha_1\alpha_2{}^2 a_2 - a_2\alpha_1 + \alpha_2 a_1)x_1}{2\alpha_1(\alpha_2{}^2+1)} - \frac{a_1(\alpha_1\alpha_2-1)(\alpha_1\alpha_2+1)x_2{}^2}{2\alpha_1(\alpha_2{}^2+1)x0} - \frac{(\alpha_1{}^2+1)\alpha_2 a_1 x_2 x_3}{\alpha_1(\alpha_2{}^2+1)x_0} + \frac{\alpha_2 d_2 x_2}{\alpha_2{}^2+1} + 2\frac{\alpha_2{}^2 d_2(\alpha_1{}^2+1)x_3}{(\alpha_1^2-\alpha_2^2)(\alpha_2{}^2+1)} \quad (8)$$

Back substitution into H_1 results in only one quadratic equation in each set. Applying the dual quaternion conjugation and the T_{DH} transformation yields exactly the equation $\overline{H}_1$ describing the rotation capability of the inverted chain and this equation is identical to the equation H_2 obtained by performing the LIA on PRR-chain

$$\begin{aligned} H_1 &: (\alpha_1{}^2\alpha_2{}^2-1)(x_1{}^2+x_2^2)+2(\alpha_1{}^2\alpha_2+\alpha_2)(x_2x_3+x_1)+(\alpha_1{}^2-\alpha_2{}^2)(x_3{}^2+1)=0 \\ \overline{H}_1 = H_2 &: (\alpha_1{}^2\alpha_2{}^2-1)(x_1{}^2+x_2^2)-2(\alpha_2{}^2\alpha_1+\alpha_1)(x_2x_3-x_1)+(\alpha_2{}^2-\alpha_1{}^2)(x_3{}^2+1)=0 \end{aligned} \quad (9)$$

Fig.2 shows the two quadrics H_1 and H_2 and one point and its inverse connected by a mirroring line colored in blue for the parameter values $a_2 = -5, a_1 = -7, d_2 = -3, d_1 = 0, \alpha_1 = -3/2, \alpha_2 = -7/5$ (α_i denote the algebraic values of the angles).

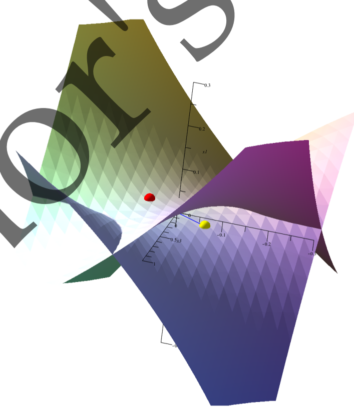

Fig. 2 Quadrics, points and connection line

6 Conclusion

In the paper it was shown that the constraint equations of an inverted kinematic chain can be obtained by simply applying quaternion conjugation to the constraint equations of the original chain. The effect of quaternion conjugation in the kinematic image space was shown to be a reflection with respect to a line and a five- dimensional subspace. Furthermore it was explicitly shown, that quaternion conjugation and the much more complicated linear implicitization algorithm yield the same result. Those displacements that are fixed in conjugation are on the intersection of a five dimensional space $x_0 = y_0 = 0$ with the Study quadric. The provided algorithm can be applied not only to any thinkable kinematic chain but also to any mechanism and its inverted which are composed of (even different) kinematic chains.

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