Efficient Computation of the Workspace Boundary, its Properties and Derivatives for Cable-Driven Parallel Robots

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Abstract. The workspace is an important property in the design of every cable-driven parallel robot. As the workspace is a complicated geometric object, it is difficult to describe changes in shape and size of the workspace when varying the design parameters of the robot. In this paper, we present an efficient method called *differential workspace hull* to describe and compute the workspace properties. The method is based on a triangulation of the surface of the robot's workspace. Furthermore, we establish an algorithm that allows to compute the influence of small changes in the design parameters on the workspace shape. A numerical example underlines the computational efficiency and accuracy of the presented method.

Key words: cable-driven parallel robots, workspace boundary, differential, parameter design

1 Introduction

The workspace $\mathscr{W}$ of a robot is the set of all poses that may be generated by this robot. For analysis and application planning, the workspace is one of its main characteristics. As the general workspace is a six dimensional volumetric object, its characterization is difficult. Merlet [5] introduces a couple of concepts to formulate meaningful descriptions such as the *constant orientation workspace* $\mathscr{W}_{\mathrm{CO}}$ or the *total orientation workspace* $\mathscr{W}_{\mathrm{TO}}$. These workspaces are three-dimensional subsets of the general workspace and can be represented as geometric objects e.g. in CAD software and stored using conventional file formats.

The determination of the workspace for cable-driven parallel robots attracted some attention. The key workspace criterion for a cable robot is its ability to control the mobile platform with positive tension in the cables which is called *wrench-feasibility* [13, 3] and to exert wrenches with the end-effector [1]. Other restrictions include the consideration of limited capabilities of the actuators in terms of velocities and accelerations as well as the avoidance of cable-cable collisions [7] and singularities.

Different methods were proposed to compute the workspace of cable robots. Interval analysis allows to make volumetric computation of the workspace. Bruckmann [2] developed an interval test for wrench-feasibility allowing for a guaranteed and continuous workspace computation. Gouttefarde [4] uses interval analysis to determine the wrench-feasible workspace. The interval algorithms are rigorous in numerical evaluation but unsensitive to small changes in its input parameters.

As a way of pragmatic representation of the workspace, one can consider only the surface or boundary of $\mathscr{W}_{\text{CO}}$ and $\mathscr{W}_{\text{TO}}$. Thus, its geometric representation is a surface in space and a convenient representation is triangulation [8]. An analytical formula for the determination of the boundary of the workspace is presented by Verhoeven [12] and it is found to be a system of univariate polynomial inequations. Unfortunately, the general expression is so complex that it seems out of reach to deal with these equations even when using advanced computer algebra systems. However, it provides insight into the structure of the wrench-closure workspace boundary that consists of pieces of polynomial surfaces with degree n where n is the degree-of-freedom of the robot. Recently, this structure is exploited in a symbolic-numeric scheme to identify its components [11] and Merlet presented a similar approach for the wrench-feasible workspace of suspended robots [6]. Both results motivate this work to employ an approximation through triangles.

2 Numeric determination of the workspace boundary

In this paper, we recall a method to compute the boundary of the workspace based on triangulation [8]. It is assumed that a compact region of the workspace is sought where disconnected other regions are neglected. This assumption is justified for the design of cable robots where one usually seeks robots with a compact and connected workspace. The computation of the workspace boundary aims at speed and precision rather than rigorous results or insight into the mathematical structure of the workspace.

In the following, we assume that a quality index is used in terms of a function $g(\mathbf{r},\mathbf{R})$. The function g yields a positive value if the pose described by the position vector $\mathbf{r}$ and the orientation matrix $\mathbf{R}$ of the mobile platform belongs to the workspace. For cable robots, such functions can be implemented by testing if the pose is *wrench-feasible* e.g. by computing a force distribution [10, 9]. If the workspace test yields a Boolean result, true is associated with a value of 1 and false is interpreted as -1.

The workspace for a given orientation of the cable robot is represented by a triangulation of its boundary. The idea for the determination of the workspace is to start with a unity sphere around the estimated center $\mathbf{m}$ and to successively extend this sphere in radial direction. Clearly, this assumption may lead to an underestimation of the workspace and the estimation depends on the chosen value of $\mathbf{m}$. The surface of the sphere is approximated by triangles which are created from n_{S} iterative subdivisions of the faces of an octahedron [8].

By making iterative subdivisions from the triangular faces of this octahedron, two sets are derived. Firstly, the set of the vertices $\mathscr{V} = \{\mathbf{v}_1,\ldots,\mathbf{v}_{n_{\text{V}}}\}$ of the tri-

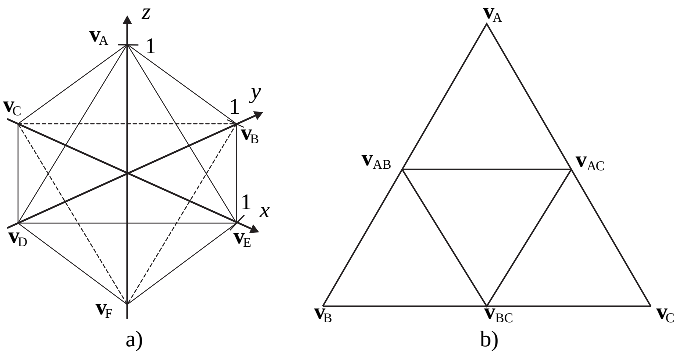


Fig. 1 a) Unit octahedron b) subdivision step for triangles

angular mesh, and the set of triangles $\mathscr{L} = \{F_1, \dots, F_{n_T}\}$ with triples of vertices indicating which triples form a triangle of the mesh. Each triangle has the form $F_i = (\mathbf{v}_{3i+1}, \mathbf{v}_{3i+2}, \mathbf{v}_{3i+3})$. Thus, we have a set $\mathscr{L}$ containing $n_T = 2^{2n_S+3}$ triangles.

In the second step, the vertices of the triangles are projected onto the boundary of the workspace. Starting from the estimated center $\mathbf{m}$ of the workspace, the line

$$L_i : \mathbf{v}_i^{(h)} = \mathbf{m} + \lambda_i \mathbf{v}_i \quad \lambda_i \in [0 \dots r_{max}] \tag{1}$$

is searched for the boundary of the workspace, which is defined by a given maximum search range r_{max}. For each position $\mathbf{r}$ generated by the iterations of the line search, we can compute an arbitrary workspace criterion such as wrench-closure, wrench-feasibility, reachability, intersection, or feasible deflection using the function $g(\mathbf{r}, \mathbf{R})$. The numerical results presented in this contribution are computed with a *regula falsi* method as it is simple and efficient for Boolean criteria. If the workspace criterion evaluates to a continuous function, methods such as Newton iteration can speed up the computation of the roots of the function. If multiple roots are found, the smallest λ_i is a conservative value for the boundary. Also rigorous search methods such as interval analysis can be used to find the first root of the workspace criterion. Furthermore, sampling or interval evaluation of a set of orientations $\mathscr{R} \subset SO_3$ allows to generalize the method to compute the total orientation workspace (see Sec. 2.1).

Finally, one ends up with the vertex $\mathbf{v}_i^{(h)} = \mathbf{m} + \lambda_i \mathbf{v}_i$ approximating the hull of the workspace with an accuracy ε_L. The corresponding triangles are rendered into a new set $\mathscr{L}^{(h)}$. Such data can be easily stored in a file such as stereo-lithography data file format (STL) or virtual reality modeling language (VRML) according to ISO 14772 which can be loaded and visualized with most CAD tools.

The rational behind this generation of the triangular grid is to separate the structure of the grid from the actual geometry. Having generated the directions $\mathbf{v}_i$ for the line search, one can store the direction and the length λ_i of these vertices in separated data structure. The direction vectors $\mathbf{v}_i$ can be pre-computed as a grid of a given resolution (i.e. iteration depth n_S). Thus, if the robot undergoes small changes in its geometry, one can re-compute the length of its vertices λ_i and perform a one-to-one comparison to the values of the original robot.

2.1 Boundary computation for different types of workspace

Having defined the data model and search strategy, one can compute the different types of the workspace. The strategy described above is straightforward to use for computing the constant orientation workspace $\mathscr{W}_{\text{CO}}$ simply by setting one specific orientation $\mathbf{R}$ for the platform. If one is interested in the maximum workspace $\mathscr{W}_{\text{max}}$, one has to modify the evaluation of the function $g(\mathbf{r},\mathbf{R})$. A position is said to belong to the maximum workspace $\mathscr{W}_{\text{max}}$, if any one orientation in a set $\mathscr{R} = \text{SE}_3$ belongs to the workspace. Thus, in the performance criterion, the function $g(\mathbf{r},\mathbf{R})$ tests a discrete grid or an interval range of orientations to be checked after the other, until an orientation is found that belongs to the workspace or until all are found to be invalid. In this case, the $g(\mathbf{r})$ is treated to be valid. This can be understood as a Boolean disjunction (logical: or) between the evaluation of all $g(\mathbf{r},\mathbf{R}), \mathbf{R} \in \text{SE}_3$. If no orientation was found to be valid, then the pose and thus $g(\mathbf{r})$ is invalid.

Computing valid positions for the total orientation workspace $\mathscr{W}_{\text{TO}}$ is done respectively but instead of searching for at least one entry in a subset $\mathscr{R} \subset \text{SO}_3$ where the workspace test is valid, one cancels the test if one element fails. In this case, $g(\mathbf{r})$ evaluates to invalid for that position. In contrast, successfully completing the full list $\mathscr{R}$ evaluates to valid. This is equivalent to the Boolean conjunction (logical: and) of all single tests $g(\mathbf{r},\mathbf{R}), \mathbf{R} \in \mathscr{R}$.

2.2 Computing properties of the workspace from the boundary

The triangulated boundary allows for geometric characterizations of the workspace. It is straightforward to calculate the surface $S(\mathscr{W})$, the volume $V(\mathscr{W})$, and the center of gravity $\mathbf{c}(\mathscr{W})$ of the workspace from the vertices as follows

$$S(\mathscr{W}) = \frac{1}{2}\sum^{\mathscr{L}} ||(\mathbf{v}_{\text{A}} - \mathbf{v}_{\text{B}}) \times (\mathbf{v}_{\text{A}} - \mathbf{v}_{\text{C}})||_2 \tag{2}$$

$$V(\mathscr{W}) = \frac{1}{6}\sum^{\mathscr{L}} ((\mathbf{v}_{\text{A}} - \mathbf{m}) \times (\mathbf{v}_{\text{B}} - \mathbf{m})) \cdot (\mathbf{v}_{\text{C}} - \mathbf{m}) \tag{3}$$

$$\mathbf{c}(\mathscr{W}) = \frac{1}{4V(\mathscr{W})}\sum^{\mathscr{L}} (\mathbf{v}_{\text{A}} + \mathbf{v}_{\text{B}} + \mathbf{v}_{\text{C}} + \mathbf{m}) \quad . \tag{4}$$

For the volume, one can find a convenient shortcut if one substitutes $\mathbf{v}_i - \mathbf{m} = \lambda_i \mathbf{u}_i$ in the parametric form with the direction $\mathbf{u}_i$ and its length from the line search λ_i. Then, the equation for the volume becomes

$$V(\mathscr{W}) = \frac{1}{6}\sum^{\mathscr{L}} \lambda_A\lambda_B\lambda_C(\mathbf{u}_A \times \mathbf{u}_B)\cdot\mathbf{u}_C \quad , \tag{5}$$

where the scalar value of the product $(\mathbf{u}_A \times \mathbf{u}_B)\cdot\mathbf{u}_C$ is equal for all triangles and depends only on the number of subdivisions n_S done. This simplification holds true if a regular solid such as the octahedron is used to generate the grid. Thus, one finds the simple form

$$V(\mathscr{W}) = \frac{(\mathbf{u}_A \times \mathbf{u}_B)\cdot\mathbf{u}_C}{6}\sum^{\mathscr{L}} \lambda_A\lambda_B\lambda_C \quad , \tag{6}$$

with the constant factor $V_i^{(n_S)} = (\mathbf{u}_A \times \mathbf{u}_B)\cdot\mathbf{u}_C$.

The accurate determination of these numbers is useful for designing of cable robots, especially if one wants to take derivatives of these indices. For computing derivatives (see Sec. 2.3), one can seldom compute the expressions in closed-form. If one has to rely on numerical approximation through finite differences, the computation for neighboring values should be as accurate as possible. Therefore, one has to balance the accuracy used in the line search with the step width of the finite difference such that the results are meaningful.

2.3 Differential hull

When analyzing the workspace of a cable robot, an interesting aspect is how the workspace depends on the geometrical and technical parameters or more generally speaking how it depends on the assumptions made and the algorithm settings. In general, the computed workspace will be changed if the parameters are differentially altered. Therefore, doing a sensitivity analysis on the parameters influencing the result of the workspace computation is revealing and can be done efficiently based on the workspace hull model proposed above. Mathematically speaking, one may ask for the derivations of the workspace caused by infinite changes of the describing parameters such as the positions of the winches $\mathbf{a}_i$, the geometry of the platform $\mathbf{b}_i$ or the feasible forces in the cables f_{min}, f_{max} (see also Tab. 1). One may also ask for the sensitivity of the constant orientation workspace $\mathscr{W}_{CO}$ for changes in the orientation $\mathbf{R}_0$. Since the workspace is a continuous set, the changes in shape and size mainly happen on its boundary. Here, the possibility is neglected that the parameter change generates a hole in the workspace which would change the workspace's topological structures. Therefore, the change in the parameters will only influence the hull of the workspace. As we have already seen when computing the workspace, it is difficult to find a closed-form solution of the workspace, hence, for computation we cannot compute the derivations symbolically. Clearly, numerical approximation using finite differences is a possible way. If we compute the workspace using discretization or interval techniques, the solution is insensitive in terms of small changes in the parameters unless one uses very small thresholds for the discretization. This problem applies both for simple discretization as well as for interval analysis. In contrast, the approximation of the workspace boundary through the hull algorithm separates the

Table 1 Overview of the parameters to be studied with the differential hull.

Geometry	Technology	Algorithm setting
proximal anchor points $\mathbf{a}_i$	cable force limits $f_{\min}, f_{\max}$	settings of the force distribution algorithm (e.g. max. iterations)
distal anchor points $\mathbf{b}_i$	cable length limits $l_{\min}, l_{\max}$	platform orientation $\mathbf{R}$ (for constant orientation workspace)
pulley radius r_{R}	applied wrench $\mathbf{w}_{\mathrm{P}}$	
	maximum cable velocity $\dot{l}_{\max}$	
	maximum cable acceleration $\ddot{l}_{\max}$	

granularity of the used grid from the accuracy in the computation. Once the number of triangles is chosen, the points on the hull can be efficiently computed with high accuracy yielding sensitivity to parameter changes.

If we now consider small changes in the geometry of the robot, we can accurately track the change in the workspace boundary with moderate computational burdens. To better understand the approach, it is important to note that the steps for determining the search directions $\mathbf{v}_i$ for the hull determination can be completed before computing the values for λ_i for each vertex. Therefore, one changes the robot model by an increment Δp and compute the resulting value for λ_i'. A suitable approximation is

$$\delta_{\mathrm{p}} \approx \frac{\lambda_i' - \lambda_i}{\Delta p} \quad . \tag{7}$$

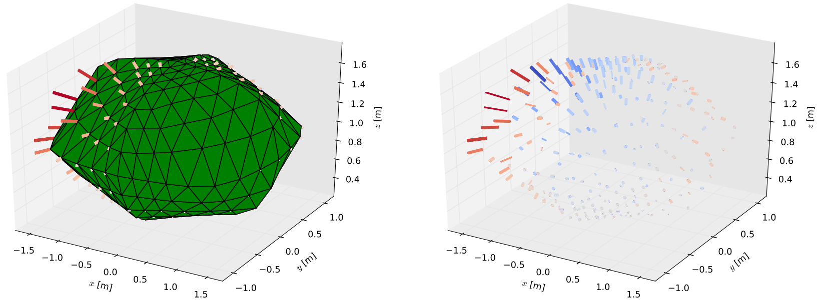
The concept of the differential workspace can be extended to compute the influence of the parameters on the shape of the workspace, i.e. to compute the derivatives of the vertices of the workspace or on the derive of the properties surface $S(\mathscr{W})$, volume $V(\mathscr{W})$, center of gravity $c(\mathscr{W})$ of the workspace, i.e. through finite differences, one computes the derivation or sensitivity

$$\frac{\partial S(\mathscr{W})}{\partial p} \quad , \tag{8}$$

where p is any numerical parameter of the geometry of the robot, the robot's technical parameters, or an algorithm parameter. An overview of parameters applicable for the sensitivity analysis is given in Tab. 1.

3 Results

In this section, an example of the differential hull approach is presented. The case study is based on the IPAnema 1 robot and the differential hull is computed for the partial derivations of the workspace hull for changes in the x-component of the first proximal anchor point $\mathbf{a}_1$. Using the differential hull, the change in the shape and size of the workspace is determined. Therefore, the algorithm computes a finite difference approximation for the differential



a) hull visible b) hull invisible

Fig. 2 Differential hull of the constant orientation workspace $\mathscr{W}_{\mathrm{CO}}$ of the IPAnema 1 robot computed with closed-form method for a finite difference in the x-component of the first proximal anchor point $\mathbf{a}_{i\mathrm{x}}$. The magnitude of the differences is amplified to make the effect of the change visible. a) The plot shows the hull with the normal lines indicating magnitude and sign of the finite difference. b) The same analysis but with an invisible hull.

$$\delta_{a_{1\mathrm{x}}} = \frac{\partial \mathscr{W}(a_{1\mathrm{x}})}{\partial a_{1\mathrm{x}}} \quad , \tag{9}$$

where the differences in the vertex $\mathbf{v}_i$ are actually expressed as differences $\mathrm{d}\lambda_i$ in the length of vertex $\mathbf{v}_i$. The results are visualized in Fig. 2. Red lines in the diagrams indicate regions with positive values of the derivatives $\mathrm{d}\mathbf{v}_i$ and thus a growth in the workspace. In contrast, blue lines represent negative derivatives which correlate with a local decrease in workspace volume. In Fig. 2b, the same results are shown in order to highlight the region with negative derivatives that are occluded by the hull in the left plot since the negative derivatives are pointing inwards from the surface of the workspace. To compute the hull, the threshold for the line search is $\varepsilon = 10^{-6}$ and the finite difference in $\mathbf{a}_{1\mathrm{x}}$ was $\Delta \mathbf{a}_{1\mathrm{x}} = 10^{-3}$. The absolute values of the finite differences range between -0.001418 and 0.001277 which indicates at maximum a one-to-one relation between the changes in the geometry and the changes in the workspace.

The computation of the differential hull is very fast; the determination of the case study took around 30 ms on a Core i5-3320M with 2.6 GHz. Therefore, all partial derivatives of the workspace volume, surface, and bounding box can be determined in less than one second making the evaluation of these differences an interesting tool for the design of cable robots.

4 Conclusions

In this paper, we proposed a scheme to compute triangulations of the constant orientation workspace as well as of the total orientation workspace for cable-driven parallel robots. The presented form for this triangulation allows to determine properties such as volume and surface both in a fast and accurate way. As changes in length of the vertices are sensitive to small changes in the geometry of the robots, it is proposed to numerically estimate the derivatives of the workspace geometry with respect to changes in the geometry parameters. This presents a useful tool in the design procedure of cable robots as one can establish relations between geometry and robot properties to perform targeted manipulation of the geometry, e.g. to determine geometric parameters that lead to a local growth of the workspace.

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