# Intrusion, Proximity & Stationary Distance

Paul Zsombor-Murray

*Dept. of Mech. Eng. & Centre for Intelligent Machines - McGill University 817 Sherbrooke St. W. Montreal QC Canada H3A 0C3, e-mail: paul@cim.mcgill.ca*



**Abstract.** Computation of intersection of right truncated cylinders of revolution and stationary distances, including least and greatest, between conics and quadrics will be re-examined using classical geometry. Solutions are provided by formulating simultaneous polynomial constraint equations that represent 3D surfaces. Previous investigations in this regard claim that the work is useful in preventing interference between rigid bodies in joint articulated mechanical systems. No such claim is made herein. Indeed the intent was to have fun by indulging in elementary "geometric thinking".

**Key words:** rigid body, collision, conics, quadrics, shortest distance

## 1 Introduction

A great deal has been written about this topic. Not long ago Agarwal, Srivatsan and Bandyopadhyay [1] published the definitive article. There is little that I can add to this and to the literature mentioned in its comprehensive bibliography. Rather I will concentrate on some of the piecemeal sub-problems and expose some not-widely-known, possibly novel, methodology.

- Since our cylinders $k_P, k_Q$ are sectioned by axis-normal planes let us represent a pair by their centreline end points $A\{1 : a_1 : a_2 : a_2\}, B\{1 : b_1 : b_2 : b_3\}$ and $C\{1 : c_1 : c_2 : c_3\}, D\{1 : d_1 : d_2 : d_3\}$ and respective radii $r, s$.
- A key sub-problem is to find on the cylinder axes, lines $\mathscr{P}$ and $\mathscr{Q}$, their common normal end points $P$ on $AB$ and $Q$ on $CD$. The closest distance between surfaces, if lengths are indefinite, is simply $|PQ| - r - s$. Line geometry will be applied.
- To find if an end disc intersects another, these are represented, *e.g.*, the one of four on $A$, by sphere $k_A : (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 - r^2 = 0$ and plane with coordinates $a\{A_0 : b_1 - a_1 : b_2 - a_2 : b_3 - a_3\}$ [1]. Contact or intrusion occurs if the line of intersection between the two planes intersects *both* spheres on real points.

[1] $A_0 = -a_1(b_1 - a_1) - a_2(b_2 - a_2) - a_3(b_3 - a_3)$

- To find if an end disc intrudes into a cylinder flank, say, $k_{a''} = k_{A''} \cap a''$, and that of $k_{Q''}: x_2^2 + x_3^2 - s^2 = 0$. $('')$ indicates all three elements are displaced so $C$ is on the origin and $D$ on the $x_1$ axis. Then the four points $X(x_1, x_2, x_3)$ of intersection of $a'' \cap k_{A''} \cap k_{Q''}$, if real, are checked to see if $0 \le x_1 \le |CD|$.
- Finally a line geometric approach to finding the octic univariate that describes stationary distances between a pair of spatial circles will be described. One of these is the shortest. Distance criteria were used by Agarwal *et al* [1] to avoid collision. My three sub-problem collision, as opposed to their four sub-problem distance, method seems simpler and sufficiently secure if actual cylindrical pieces are buffered by increase in length and radius.

## 2 Common Normal Cylinder Centreline End Points

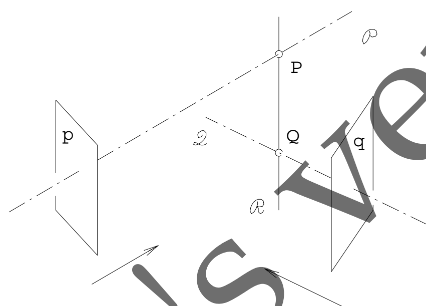



**Fig. 1** Common Normal $\mathscr{R}$

Cylinder centrelines $\mathscr{P}_r, \mathscr{Q}_r$ are represented by their *radial* Plücker coordinates directly computed with point pairs $A, B$ and $C, D$.

$$\mathscr{P}_r\{p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}\},\ \ \mathscr{Q}_r\{q_{01} : q_{02} : q_{03} : q_{23} : q_{31} : q_{12}\}$$

Pencils $p, q$ of planes normal to $\mathscr{P}, \mathscr{Q}$ respectively are used to define *axial* line $\mathscr{R}_a$.

$$p\{P_0 : p_{01} : p_{02} : p_{03}\},\ \ q\{Q_0 : q_{01} : q_{02} : q_{03}\}$$

$P_0$ and $Q_0$ are the two unknowns necessary to find end points $P$ and $Q$ of common normal axial line $\mathscr{R}_a$ on lines $\mathscr{P}_r$ and $\mathscr{Q}_r$ using intersections

$$\mathscr{R}_a\{R_{01} : R_{02} : R_{03} : R_{23} : R_{31} : R_{12}\},\ \ \mathscr{P}_r \cdot \mathscr{R}_a = 0,\ \ \mathscr{Q}_r \cdot \mathscr{R}_a = 0$$

$P = p \cap \mathscr{P}_r$ and $Q = q \cap \mathscr{Q}_r$, *e.g.*, $p_i = \sum_{j=0}^{3} p_{ij} P_j$ thus, where $P_j = p_{0j}$.

$$\begin{array}{llll} p_0 = & & p_{01}P_1 & +p_{02}P_2 & +p_{03}P_3 \\ p_1 = & -p_{01}P_0 & & +p_{12}P_2 & -p_{31}P_3 \\ p_2 = & -p_{02}P_0 & +p_{12}P_1 & & +p_{23}P_3 \\ p_3 = & -p_{03}P_0 & +p_{31}P_1 & -p_{12}P_2 & \end{array} \tag{1}$$

If $|PQ| - r - s \leq 0$ to establish collision we check that $P$ is on either or between $A$ and $B$ and that $Q$ bears similar relation to $C$ and $D$. This can be done, *e.g.*, directly with $A_0$ [2] and $B_0$, the constant coefficients of equations of normal planes on $A$ and $B$, by verifying that $A_0 \leq P_0 \leq B_0$ or $A_0 \geq P_0 \geq B_0$.

## 3 Collision or Intersection of Cylinder Ends

To check if cylinder ends on points, say, $A, C$ interfere we apply Eqs. 2.

$$\begin{aligned} a: &\ A_0 + A_1x_1 + A_2x_2 + A_3x_3 = 0 \\ c: &\ C_0 + C_1x_1 + C_2x_2 + C_3x_3 = 0 \\ k_A: &\ (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 - r^2 = 0 \\ k_C: &\ (x_1 - c_1)^2 + (x_2 - a_c)^2 + (x_3 - c_3)^2 - s^2 = 0 \end{aligned} \tag{2}$$

Consider Fig. 2. Cylinder end discs will have a line segment, or at least a point, in common if simultaneous solution of the first three of Eqs. 2 and the the first two and the last *both* yield real $X$ at $P$ and $Q$. Note how descriptive geometry and judicious choice of view pair provide clear visualization of the process.

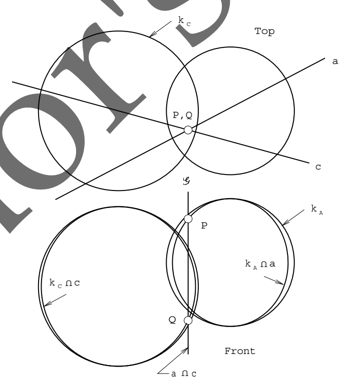



**Fig. 2** Line on Cylinder End Planes Intersect both Spheres

[2] In this case, as opposed to that mentioned in the introduction, $A_0 = -p_{01}a_1 - p_{02}a_2 - p_{03}a_3$

## 4 Collision or Intersection of a Cylinder Surface with an End

First the cylinder, radius $s$, with ends on $C, D$ is displaced so $C$ is on origin $O$ and $D$ is along $x_1$-axis, $x_1 > 0$. Then the translation $C \rightarrow O$ is imposed upon plane $a$ and centre $A$ of sphere $k_A$ followed by the rotation necessary to make $A \rightarrow B$ parallel to $x_1$-axis. So $a, A, k_A \rightarrow a', A', k_{A'} \rightarrow a'', A'', k_{A''}$.

### 4.1 Translation

$$A \rightarrow A': \begin{bmatrix} 1 & 0 & 0 & 0 \\ c_1 & 1 & 0 & 0 \\ c_2 & 0 & 1 & 0 \\ c_3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ c_1 + a_1 \\ c_2 + a_2 \\ c_3 + a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} \tag{3}$$

Although the translation of point $A$ via Eq. 3 is obvious, plane coordinates, being of dual species, are transformed by the *cofactor* of the translation matrix as in Eq. 4.

$$a \rightarrow a': \begin{bmatrix} 1 & -c_1 & -c_2 & -c_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} A_0 - c_1 A_1 - c_2 A_2 - c_3 A_3 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} A'_0 \\ A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} \tag{4}$$

### 4.2 Normed Quaternion and Rotation Matrix

The normed quaternion $\mathbf{v}$ or rotation matrix $[\mathbf{V}]$ that rotates direction $C \rightarrow D$ as required must premultiply $A', a'$. A neat property of $[\mathbf{V}]$ is that it is identical to its cofactor. $\mathbf{v}$ and $[\mathbf{V}]$ are introduced in Eq. 5.

$$\begin{aligned} \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \cos(\phi/2) \\ \cos\alpha \sin(\phi/2) \\ \cos\beta \sin(\phi/2) \\ \cos\gamma \sin(\phi/2) \end{bmatrix}, \ [\mathbf{V}] = \begin{bmatrix} r_{00} & 0 & 0 & 0 \\ 0 & r_{11} & r_{12} & r_{13} \\ 0 & r_{21} & r_{22} & r_{23} \\ 0 & r_{31} & r32 & r_{33} \end{bmatrix} \\ = \begin{bmatrix} v_0^2 + v_1^2 + v_3^2 + v_3^2 & 0 & 0 & 0 \\ 0 & v_0^2 + v_1^2 - v_2^2 - v_3^2 & 2(v_1 v_2 - v_0 v_3) & 2(v_1 v_3 + v_0 v_2) \\ 0 & 2(v_2 v_1 + v_0 v_3) & v_0^2 - v_1^2 + v_2^2 - v_3^2 & 2(v_2 v_3 - v_0 v_1) \\ 0 & 2(v_3 v_1 - v_0 v_2) & 2(v_3 v_2 + v_0 v_1) & v_0^2 - v_1^2 - v_2^2 + v_3^2 \end{bmatrix} \end{aligned} \tag{5}$$

Elements $v_i$ of a normed quaternion are also called Euler-Rodrigues parameters. $[\cos\alpha \ \ \cos\beta \ \ \cos\gamma]^\top$ is the unit vector –expressed in terms of direction cosines– in the direction of the rotation axis while $\phi$ is the rotation angle in a right-hand screw sense. To get quaternion from rotation matrix –except for half-turns which I won't mention here– we use the diagonal elements $r_{ii}$ to get $v_i^2$ as shown in Eq. 6.

$$\begin{bmatrix} r_{00} & 0 & 0 & 0 \\ 0 & r_{11} & r_{12} & r_{13} \\ 0 & r_{21} & r_{22} & r_{23} \\ 0 & r_{31} & r_{32} & r_{33} \end{bmatrix} \rightarrow \begin{array}{l} v_0^2 = (r_{00} + r_{11} + r_{22} + r_{33})/4 \\ v_1^2 = (r_{00} + r_{11} - r_{22} - r_{33})/4 \\ v_2^2 = (r_{00} - r_{11} + r_{22} - r_{33})/4 \\ v_3^2 = (r_{00} - r_{11} - r_{22} + r_{33})/4 \end{array} \tag{6}$$

### 4.3 Rotation

The rotation sought turns $C'D'$, $C' \equiv C'' \equiv O$, onto the $x_1$-axis through rotation through $\phi$ about $O$ via unit vector $\mathbf{n} = [n_1 \;\; n_2 \;\; n_3]^\top$ into $\mathbf{x} = [1 \;\; 0 \;\; 0]^\top$. The unit vector $\rho$ in the rotation axis direction is given by Eqs. 7.

$$\begin{gathered} \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{\sqrt{(d_1-c_1)^2+(d_2-c_2)^2+(d_3-c_3)^2}} \begin{bmatrix} d_1 - c_1 \\ d_2 - c_2 \\ d_3 - c_3 \end{bmatrix} \\ \rho = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = \frac{\mathbf{n}\times\mathbf{x}}{|\mathbf{n}\times\mathbf{x}|} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} / |\mathbf{n}\times\mathbf{x}| = \frac{1}{\sqrt{n_2^2+n_3^2}} \begin{bmatrix} 0 \\ n_3 \\ -n_2 \end{bmatrix} \end{gathered} \tag{7}$$

To complete the computation of the quaternion elements *cum* Euler-Rodrigues parameters we need $\cos(\phi/2)$ and $\sin(\phi/2)$. Imagine vectors $\mathbf{n}$ and $\mathbf{x}$ placed tail-to-tail on $O$, a line segment joining their tips, its mid-point $M$, the tip of vector $\mathbf{m}$ from $O$. Consider that $|\mathbf{m}| = \cos(\phi/2)$ and $|\mathbf{x} - \mathbf{m}| = \sin(\phi/2)$. All this is illustrated in Fig. 3.

$$\cos\frac{\phi}{2} = \frac{1}{2}\sqrt{(1+n_1)^2 + n_2^2 + n_3^2}, \; \sin\frac{\phi}{2} = \frac{1}{2}\sqrt{(1-n_1)^2 + n_2^2 + n_3^2} \tag{8}$$

As an exercise the reader may reformulate the problem of Eq. 9 as $a \cap k_A \cap k_Q$ by

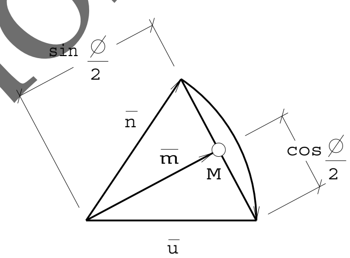



**Fig. 3** Rotation and Significance of Half-Angle Sine and Cosine

displacing $k''_Q \rightarrow k_Q$ instead of $a \rightarrow a''$ and $k_A \rightarrow k_{A''}$.

### *4.4 Constraint Equations*

The implicit equations of plane $a''$, sphere $k_{A''}$ and the cylinder $k_{Q''}$, to be solved simultaneously to yield points $X$, appear in Eqs. 9.

$$\begin{aligned} a'':\ & A''_0 + A''_1 x_1 + A''_2 x_2 + A''_3 x_3 = 0 \\ k_{A''}:\ & (x_1 - a''_1)^2 + (x_2 - a''_2)^2 + (x_3 - a''_3)^2 - r^2 = 0 \\ k_{Q''}:\ & x_2^2 + x_3^2 - s^2 = 0 \end{aligned} \tag{9}$$

Fig. 4 contains two views showing the plane $a''$ in edge or line view at upper left and

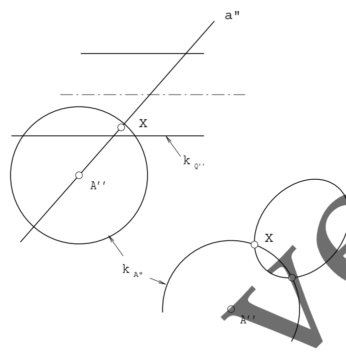



**Fig. 4** Line on Cylinder End Planes Intersect both Spheres

the circle of cylinder $k_{P''}$ circular end disc together with the elliptical plane section of cylinder $k_{Q''}$ The existence of real points $X$ indicate encroachment of the surfaces. If radius $r$ is so small as to place the disc entirely within $k_{Q''}$ without triggering the common normal length criterion this condition is checked via the distance between disc centre point $A$ and centre line $\mathscr{Q}$ on $CD$ being less than radius $s$.

## 5 Stationary Distances between Spatial Circles

In the article [1] the shortest distance between two circles is made use of to account for impending contact between cylinder end edges and an octic solution is referred to. Although the approach introduced in § **3** handles this situation automatically it is of interest to reveal how these distances can be computed using a line $\mathscr{R}$ that intersects circle axis lines $\mathscr{M}$ and $\mathscr{N}$ on respective points $M, N$. $\mathscr{R}$ will be defined by points $P, Q$ on circles $k_a$ and $k_c$, respectively, as shown in Fig. 5. Line $\mathscr{R}$, shown in Fig. 5, depicts a typical line belonging to two line congruences. One contains all lines on points on circle $k_a$ and normal to the circle tangent at that point, $P$. This property is ensured by the intersections $P \in k_a$, $P \in \mathscr{R}$, $M \in \mathscr{R}$, $M \in \mathscr{M}$, *i.e.*,

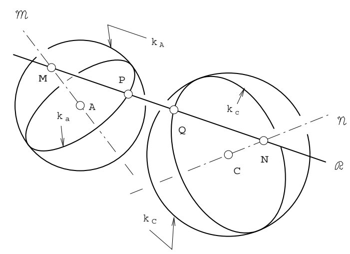
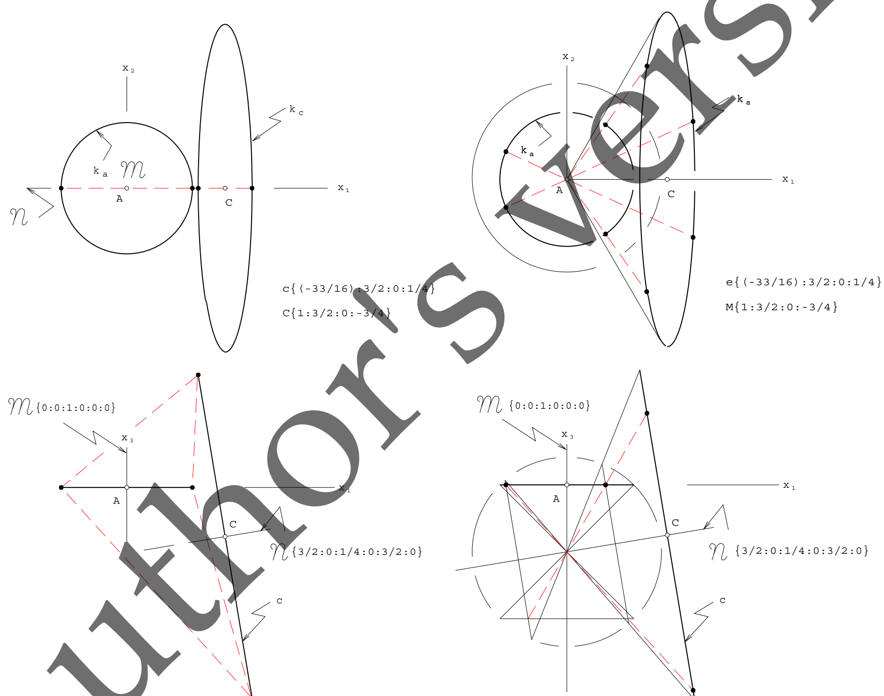



**Fig. 5** Congruence of Normal Lines on Circles

$\exists \mathscr{M} \cap \mathscr{R}$ and $\exists \mathscr{N} \cap \mathscr{R}$. The other congruence on circle $k_c$ gives rise to similar relationships, *viz.*, $Q \in k_c, Q \in \mathscr{R}, N \in \mathscr{R}, N \in \mathscr{N}$. Dissecting these relations yields six equations, Eqs. 10, in six Cartesian points coordinates, $P(p_1, p_2, p_3), Q(q_1, q_2, q_3)$.




(a) 4 Joining Lines on Plane of Axes (b) 4 Joinings Symmetric to Plane of Axes

**Fig. 6** Eight Connections between Circles

Using Gröbner basis and an arbitrary pair of spatial circles these simultaneous equations yield an octic univariate in one of the $p_i, q_i$ and the basis provides a systematic way to compute the remaining five. *I.e.*, each successive basis polynomial contains one linear unknown in terms of those already evaluated. In general, there are only four real stationary distances among the eight solutions. Are there circle dispositions that admit eight real solutions? Again, geometric thinking and descriptive geometry reveal in Fig. 6 eight connecting segments that satisfy Eqs. 10.

$$
\begin{gathered}
k_a = k_A \cap a,\ \ P \in k_a,\ \ k_c = k_C \cap c,\ \ Q \in k_c \\
A_0 + A_1 p_1 + A_2 p_2 + A_3 p_3 = 0 \\
(p_1 - a_1)^2 + (p_2 - a_2)^2 + (p_3 - a_3)^2 - r^2 = 0 \\
C_0 + C_1 q_1 + C_2 q_2 + C_3 q_3 = 0 \\
(q_1 - c_1)^2 + (q_2 - c_2)^2 + (q_3 - c_3)^2 - s^2 = 0 \\
\exists M = \mathscr{M} \cap \mathscr{R},\ \ \exists N = \mathscr{N} \cap R \\
m_{01}R_{01} + m_{02}R_{02} + m_{03}R_{03} + m_{23}R_{23} + m_{31}R_{31} + m_{12}R_{12} = 0 \\
n_{01}R_{01} + n_{02}R_{02} + n_{03}R_{03} + n_{23}R_{23} + n_{31}R_{31} + n_{12}R_{12} = 0 \\
\mathscr{M}_r\{A_1 : A_2 : A_3 : a_2A_2 - a_3A_2 : a_3A_1 - a_1A_3 : a_1A_2 - a_2A_1\} \\
\mathscr{N}_r\{C_1 : C_2 : C_3 : c_2C_2 - c_3C_2 : c_3C_1 - c_1C_3 : c_1C_2 - c_2C_1\} \\
\mathscr{R}_a\{p_2q_3 - p_3q_2 : p_3q_1 - p_1q_3 : p_1q_2 - p_2q_1 \\
: p_0q_1 - p_1q_0 : p_0q_2 - p_2q_0 : p_0q_3 - p_3q_0\}
\end{gathered}
\tag{10}
$$

## 6 Conclusions

Using implicit sphere, plane and cylinder equations, some geometric thinking and descriptive geometry I've tried to unify the computational sub-problems pertinent to collision and intrusion between two cylinders and use a consistent nomenclature among them. Have any special cases been overlooked? Yes, a small end disc can intrude into a large cylinder undetected. Do you see how to overcome this using sphere centre $A''$? Was this case covered in [1]? Apologies for my, in places, didactic tone. Furthermore why should I cite more than one article? If it's the right one, clutter is undesirable.

**Acknowledgements** Jean-Pierre Merlet taught me in 1995 when he was at the second CK in Milano –the first was at Schloß Dagstuhl in 1993– that if you can formulate an algebraic problem with eight solutions, an upper bound, and can construct an example, a lower bound, with that number the issue is then settled.

## References

1. Agarwal, S., Srivatsan, R.A. and Bandyopadhyay, S. : Analytical determination of the proximity of two right-circular cylinders in space. Advanced Robotics, **JMR-15-1218 Bandyopadhyay doi:10.1115/1.4032211**, 1–11 (2015)